The Signaling Problem: Using Exploding Dots to Solve an Accessible Mystery in an Elementary-Aged Math Circle

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Cover Page Footnote
I thank Rachel Steinig for assisting in my research by writing a blog report about part of session four in the course described in this article and for taking notes and capturing student quotes during all six sessions. Rachel also assisted in leading the sessions whenever extra facilitation was needed. I thank Robert Kaplan (The Global Math Circle) and Ellen Kaplan for providing the Signaling Problem and for providing guidance on how to implement it. I also thank Talking Stick Learning Center for institutional support, the Mathematical Sciences Research Institute for financial support of the Talking Stick Math Circle, and Angela Hoppel and Asha Larsen for photography.

This article is available in Journal of Math Circles: https://digitalcommons.cwu.edu/mathcirclesjournal/vol1/iss1/5
The Signaling Problem: Using Exploding Dots to Solve an Accessible Mystery in an Elementary-Aged Math Circle

Rodi Steinig *

Talking Stick Learning Center

Many people want to facilitate Math Circles for younger students but do not know how. This article provides a model for how to create an engaging Math Circle for students aged 8-10 to explore different number bases and gives a detailed narrative to guide prospective instructors through the class. The narrative follows a group of eight students spending six weeks joyfully discovering underlying mathematical structure without being told what to do.

**Keywords:** Cryptology, Exploding Dots, Navajo Code Talkers, Pattern seeking, Signaling Problem

1 Introduction

The Signaling Problem is an open-ended, multi-level mathematical problem that presents students with an accessible mystery. It can be solved in binary using Exploding Dots, which allows it to serve as a vehicle for exploring deep ideas in mathematics. This article provides a written description of the activity as well as a successful implementation, which can serve as a guide for Math Circle leaders, especially those who are just beginning to teach younger students. At the end, readers are invited to reflect on whether the implementation lived up to the pedagogical paradigm developed by Math Circle pioneers Robert and Ellen Kaplan.

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1.1 Pedagogical Goals

There are many different types of Math Circles. According to Robert Kaplan, in The Global Math Circle,

We never teach anyone anything. It is a school without lectures, without textbooks, without grades, without tests, without homework. And what does it have left? Math. And the beauty of it. And collegial conversation. Never competitive... A very small group of people and a leader who puts an accessible mystery in front of them. An accessible mystery: something which really leads behind appearances to the underlying structure... Hidden structure, which takes a lot of imagination, goodwill, conversation with one another to uncover, to come to grips with, to still have doubts at the end. But informed doubts. [8, 00:10]

Kaplan instructs students to solve problems using “the Math Circle approach, where we do math in the real way, with you guys doing the thinking. So you own it. It’s yours.” In other words, says Kaplan, “Tell me and I forget. Ask me and I discover” [8, 44:10]. Other key components of this pedagogical model include:

- the practice of allowing and encouraging questions, challenges, and mistakes
- the facilitator learning from the students
- the engagement in mathematics that is “joyous, uplifting, meaningful, and relevant for one and all” [16, para. 6]

In the spirit of mathematics, I will state a major assumption of this article: the assumption that this model is desirable for students. I will not get into details about the “what” and the “why” of the benefits that accrue. Let us accept those as a given.

1.2 The Problem

The Signaling Problem is a six-week Math Circle course for students ages 8-10:

You and your parents are in your coastal house at a time of great danger. A person who can end the danger is in a ship. At some point in the night, the ship’s captain hops into a rowboat and comes close enough to shore to see the upstairs windows in your house. It
is your responsibility to signal to the captain when it's safe to come ashore. The safety code is “27.” You have 5 visible windows (2 in the children’s room, 1 in the hall, and 2 in the parents’ room), 5 candles, and nothing else you can use to signal. How can you do it? [R. Kaplan, personal communication, October 30, 2012]

In level one of this multi-level problem, the people in the house have only one message to send. In level two, they have more than one. In level three, there are many messages that must be encoded. For guidance, Kaplan provides one possible solution to the problem:

Since 27 in binary is 11011 . . . [you can] leave the lights on in the bedroom windows but not the hall (having agreed with the grizzled sea captain a week before that light = 1, dark = 0), and you have the message “27” sent in binary! Not only can the enemy not unravel that, they wouldn’t even suspect that these innocent lights meant anything more than that the family was at home, reading in bed! And now, there are 31 other messages that can similarly be sent, which your kids can agree on and flash to one another (taking care that the lights read from left to right, from outside: not a problem with symmetrical 11011). [ibid]

The underlying key mathematical ideas explored in this course were cryp-tology, base systems, counting strategies, and addition. The most frequent underlying key mathematical skill explored in this course was precision: clarifying the problem, questioning everything, defining terms, and discerning between convention and truth. The students also frequently engaged in pattern seeking; the uncovering, making, and stating of assumptions; the pursuit of certainty (proof); and the forming and testing of conjectures. All the while, they practiced collaboration and the communication of mathematical ideas, two skills vital to the advancement of mathematics (both personally and professionally).

I found [17] and [1] extremely helpful in providing specific problems and activities that generate deep mathematical thinking, as well as useful in providing me with a deeper understanding of different base systems. I found [8] a great reminder of everything I love about Math Circle pedagogy. For source material for our interludes on Plato, I went right to Plato’s Republic. The most helpful versions [2, 3] had a bit of commentary introducing the dialogues. Dr. Karen Carr’s article [3] also helped me to understand the material better.

The article “The Navajo Code Talkers” describes in fascinating and clear detail the indecipherable code that Navajo Marines used in World War II to
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enable the US to defeat Japan [11]. Also, [7] gave valuable historical insights. For a fascinating and more in-depth primary source account that details the code and puts it into its full cultural and historical perspective, I found in [10] a narrative that I could not put down for days. Finally, I learned interesting facts about the king’s interest in the use of an octal system in [4].

2 Intended Implementation

The “plan” was to present the question in session one, spend some weeks letting the students explore the problem, do some math history and Exploding Dots along the way, and (ta-da!) by week six the students would come up with a solution without ever having been told that they can use Exploding Dots to solve it (Table 1). That is not exactly how it went. Nothing ever goes exactly as planned in most Math Circles. That is by design. My planning shows the organic quality of Math Circle lesson plans. Each session’s plan was created based upon what happened in the prior session. We never had time to do everything on the list, such as inverse machines, because of deep student engagement in the accessible mysteries.

I posed the problem a week before introducing Exploding Dots to facilitate the students eventually feeling a need for this method. I used the approach of posing a problem at a much higher level than the students’ comprehension and ability in order to eventually joyfully engage in the skills required for solving it. Not much can spoil most kids’ parties more than opening with the line, “Let’s talk about place value.”

I wrote an agenda for the first session and planned to adapt each session’s plan in response to the students, keeping in mind that Math Circle agendas are living things that evolve from minute to minute. You will see from Table 1 that I over-planned for the first session. This is typical. When a problem is compelling, a true accessible mystery, the students spend a very long time savoring it. After the second session, I emailed parents:

*I will tell you now what I “plan” (ha!) to cover but know that since “math is freedom” (to quote Robert Kaplan), we might not follow the plan. The basic plan will be to continue with Exploding Dots and the Signaling Problem (a more advanced version), to talk about Navajo Code Talkers, and possibly discuss another Plato anecdote.*

By session five, our penultimate session, I wanted to set the students up for success. Time to solve the Signaling Problem was running out. My plan
Table 1

Session schedule. Italics signify my post-lesson notes.

<table>
<thead>
<tr>
<th>Session</th>
<th>Planned Activity</th>
<th>Specifics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Bobblehead Doll</td>
<td>Focusing activity</td>
</tr>
<tr>
<td></td>
<td>Signaling Problem</td>
<td>Introduction of problem and more challenging version</td>
</tr>
<tr>
<td></td>
<td>Function/Inverse Machines</td>
<td>Pattern seeking</td>
</tr>
<tr>
<td></td>
<td>Navajo Code Talkers</td>
<td>Did not get to inverse machines</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ran out of time</td>
</tr>
<tr>
<td>2</td>
<td>Plato Dialogue</td>
<td>Math history (philosophy of math education)</td>
</tr>
<tr>
<td></td>
<td>Function Machines</td>
<td>Ran out of time</td>
</tr>
<tr>
<td></td>
<td>Exploding Dots</td>
<td>Students figure out how it works</td>
</tr>
<tr>
<td></td>
<td>Signaling Problem</td>
<td>Continued from session one</td>
</tr>
<tr>
<td>3</td>
<td>Signaling Problem</td>
<td>Introducing even more challenging version</td>
</tr>
<tr>
<td></td>
<td>Navajo Code Talkers</td>
<td>Math history (WWII, cryptology)</td>
</tr>
<tr>
<td></td>
<td>Exploding Dots</td>
<td>Making predictions</td>
</tr>
<tr>
<td>4</td>
<td>Exploding Dots</td>
<td>Addition</td>
</tr>
<tr>
<td></td>
<td>Plato’s Allegory of the Cave</td>
<td>Math history (nature of truth)</td>
</tr>
<tr>
<td></td>
<td>Addition</td>
<td>Methods to add three-digit numbers</td>
</tr>
<tr>
<td></td>
<td>Exploding Dots</td>
<td>Introducing antidots</td>
</tr>
<tr>
<td></td>
<td>Exploding Dots</td>
<td>Did not get to this</td>
</tr>
<tr>
<td></td>
<td>Exploding Dots</td>
<td>Base five</td>
</tr>
<tr>
<td>5</td>
<td>Write numbers on paper slips</td>
<td>Appetizer for Exploding Dots</td>
</tr>
<tr>
<td></td>
<td>Exploding Dots</td>
<td>How to write the code (Five boxes/digits parallel windows in Signaling Problem)</td>
</tr>
<tr>
<td></td>
<td>Exploding Dots</td>
<td>Change the rule to octal</td>
</tr>
<tr>
<td></td>
<td>Signaling Problem</td>
<td>Embedding Alien Problem into the Signaling Problem</td>
</tr>
<tr>
<td>6</td>
<td>Exploding Dots</td>
<td>Binary with five digits/boxes</td>
</tr>
<tr>
<td></td>
<td>Signaling Problem</td>
<td>Solving it</td>
</tr>
<tr>
<td></td>
<td>Signaling Problem</td>
<td>Acting it out</td>
</tr>
</tbody>
</table>
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became more goal-oriented and less discovery-oriented. After our fourth session, the students were not really embracing different base systems despite many rounds of Exploding Dots. Therefore, I introduced the subsidiary Alien Problem to put this mathematical idea more on the class’ radar.

By class six, I abandoned detailed planning and was ready to meet the students where they were no matter what. What strikes me when I look at my planning book for the final session’s plan is that I only wrote four words. This is the sign of a true accessible mystery, that the students are owning it so much now that lesson planning is practically unnecessary.

3 Session One

3.1 Presenting the Problem

I first engaged the nine students in a focusing activity, the Bobblehead Doll,\(^1\) to harness everyone’s energy. Then I posed the Signaling Problem outlined in Section 1.2. Immediately, the students began spurting clarifying questions (Figure 1): What is the danger? Are there any other windows? Any doors? Can you get onto the roof? How many matches do you have? What is between the house and the water? Does the enemy know the code?

![Figure 1. Students make sense of the problem (left), ask questions (middle), and generate conjectures (right). (Photos by A. Larson.)](image)

Matteo (a pseudonym, as are most), Daniel, and Juan-Carlos stood up with excitement. Alex and Joanna (and others) took over writing the problem:

\(^1\)I have written previously about how to use a bobblehead doll to focus attention and build problem-solving endurance [15]. I learned the bobblehead doll activity originally at one of the wonderful annual conferences of the Mindfulness in Education Network: http://www.mindfuled.org/conferences/.
There was a six-year-old princess who was kidnapped! Her kidnapping started a war. Enemies [on the kidnapper’s side] are surrounding your house and you cannot get out without losing your life! Spies are in the water and the bushes in front of the house!

Possible conjectures on the solution, and their flaws, started flowing:

“We could strike the match, or light the candles, 27 times,” suggested Mavis and Alex. “But you don’t know when the captain is coming,” someone countered.

“We could melt the candles and shape them into the number 27 like birthday number candles,” suggested Alex. “But the enemy can’t see the signal because then they would get prepared,” countered Cloe.

“We could make shadow puppets and form the number 27 with our hands,” suggested Daniel. This was a very popular idea, but was foiled by the idea of enemies, again.

“We could distract the enemies and jump out of the house and get a raft and go get the captain,” suggested Alex. “Spies! You can’t leave the house!” reminded Ezra.

“We could watch for the captain and see when the captain comes and then point a laser pointer into a 27,” proposed Joanna. “Too dark, and besides, spies again,” someone reminded.

Students started seeing more flaws than possible solutions (Figure 2), an important math skill. We cannot say with certainty what is possible until we know what is impossible. Ezra pointed out that “you have to know if, one, the captain is loyal, and two, that it’s really him.” This was met with discouragement until I gave the group permission to make assumptions. As the Circle evolves, they will get to a point where no permission is needed; they will develop the math skill of making assumptions and, very importantly, stating them. In other words, reports Ellen Kaplan, “It takes a while, as the Quakers say, for the meeting to mature!” [E. Kaplan, personal communication, October 8, 2012]

Our group then stated three assumptions: the captain is loyal, the person in the rowboat is truly the captain, and we cannot know for sure whether the
enemy knows the code. They formed these assumptions with my coaching. When the students discussed their uncertainties, I asked what things they want to be true as they work on the problem. I used the word assumption repeatedly in an organic way until they could not help but realize what it means. I also told them explicitly that in mathematics when (1) you are not sure whether something is true, and (2) it is impossible to find out or figure out whether it is true, and (3) you want or need it to be true to advance your problem solving, you are allowed to declare it to be true as long as you tell everyone your decision. The students debated and then came to a consensus among themselves about what their assumptions would be.

Ruby then suggested, “Since this is a Math Circle, the answer probably has to do with math.” This comment moved the discussion in another direction. Someone mentioned using a code for 27. Students wondered whether the enemy could crack the code. Matteo proposed adding in a false code. Juan-Carlos disagreed, saying, “But a false code might not necessarily be the best way to distract the enemy.” While not everyone agreed on the need to distract the enemy, everyone liked the code idea. Discussion turned to possible codes (Figure 3). Someone suggested using two candles (the number 11) to represent 27 (“safety”). Another suggestion was to value the candles in each room differently: candles in the children’s window were worth 10 and those in the
hall worth seven. Then the captain could add up the value of each lit candle to know when it is safe to bring the “VIP” (as named by Juan-Carlos) to shore.

Figure 3. Students’ devised message codes.

3.2 Follow-Up Question

The class seemed happy with this solution, so I threw a possible wrench into their plan: “There are other messages you may need to signal to the captain. What might they be?” I had prepared the follow-up question, “How many other messages can be similarly sent?” but in the moment I phrased it more specifically to capitalize on the excitement about the captain. The group suggested “SOS at the house,” “Enemy sending ships out,” and “General danger to the ship.” The kids assigned each of these messages a numeric value: 44, 13, and 16, respectively (Figure 3).

Suddenly, we were out of time; parents were arriving. As our hour was up, I stated that I was confused and needed help seeing how these numbers would work with the code. Striking a pose as a confused person moves me
toward my goal of avoiding any verbalizing that could contribute to problem-solving. People were still calling out ideas as I was trying to walk out the door: “Seagulls!” “Distractions!” “False codes!”

4 Session Two

4.1 Some Math History: Plato

Last week, Daniel had requested that we learn about Plato in this course. I knew nothing about Plato at the time, so I educated myself a bit between sessions. My assistant Rachel, who was 13 years old at the time, and I began this session dramatizing Plato’s dialogue with Glaucon on the merits of learning to calculate [12], and then discussed it with the students.

4.2 A Tentative Solution to the Signaling Problem

We then continued our Signaling Problem (Figure 4). We were essentially creating a code for a code. Students tested last week’s conjecture to see if the code worked out arithmetically. It did not. Ezra proposed a new system where each window could represent a different power of 10: units for the parents’ rooms, tens for the hall, and hundreds for the children’s rooms. This sounded so “mathematical” that once everyone understood it, they were convinced that it must work.

Sometimes mathematical language can be a barrier and can intimidate people into thinking they are dumb. In most cases, this is not intentional. Mathematical language is concise and precise, and there is a huge value in that. But I have seen it be a hindrance to learning for many. For a lot of students, when the abstract language created by others starts flowing, students’ ears and minds close. Kaplan says:

The kids should invent the notation and symbols themselves. Why use the inherited, old-fashioned, clouded symbols? Make your own. I will adapt to them. The world will adapt to them. If you do not like writing fractions this way, write them that way, or that way, or any way you choose, as long as you explain what you mean. [8, 47:21]

Despite the authoritative-sounding language, Ezra’s suggested power-often system did not work; some of the code numbers could not be expressed with only five candles. For instance, said Mavis, “The 44 wouldn’t work, but 40 would, because you don’t have a candle as small as a four.”
At this point some students wanted to revisit what I call “engineering” approaches to the problem: spies, distractions, seagulls, swimming pools, letters burnt into a sheet, and a burning banshee. The group was discouraged by the math. I insisted that we not give up on using codes just because it is challenging. I wanted them to persevere, so I focused the group on starting lists of questions and conjectures on the board.

The list worked. It prompted Juan-Carlos to posit a new conjecture based upon Ezra’s idea: “Let’s add in more numbers.” Juan-Carlos suggested that “one window could be worth three, and then you could put a six in the one remaining window. Then you could do the 16 and the 13.” The group calcu-
lated that the numbers six, seven, 10, 10, and three could be combined to make our signals 27, 16, 44, and 13, as long as we change 44 to 36.²

“Danger to the house: 36. All the windows lit up!” proclaimed Daniel triumphantly.

While some students checked these numbers, others realized that each window could be a different signal without the need to add numbers. This plan seemed to address Cloe’s concern that “We only have five candles; do we need to show all these messages at once?” and Mavis’s concern that the “6-7-10-10-3” plan would only work if the people in each room could communicate with each other. One student, so excited by this new idea, rushed up to the board and drew it.

It looked like we had two solutions: one involving adding numbers in multiple windows and another involving each single window representing a code. People wondered: “Do these both work?” We discussed, deliberated, and calculated. No one saw a problem with the solutions.

I never used the phrase “the solution;” I only said “a solution.” This is to open students’ minds up to the reality that there are most likely no problems with only one solution method.

Alex asked multiple times, “Rodi, if we see no problems with it, it must be right, isn’t it?”

“It seems to be right, but that I’m not sure,” I replied. “Something didn’t sit right with me, and I need time to think about it.”

“We don’t have to be 100% sure,” argued Alex. She expressed some (informed) doubt, but wanted to declare it solved right then.

I suggested that we switch gears and come back to the problem so that we could be 100% sure, and that there is value in taking a break, to be able to see the problem with new eyes and insights after being away from it for a bit.

²To make 16, which is the code for danger to the ship, for instance, put a candle in one of the children’s rooms, worth 10, and in the hall, worth six. Likewise, one candle lit up in every room adds up to 36.
4.3 Introducing Exploding Dots

“Ezra’s idea of using units, tens, and hundreds reminds me of a game. Have you ever played Exploding Dots?” I moved the group to the other board to demonstrate what I consider a place-value function machine. The way I do function machines is to have the students provide the input number, I provide the output number, and then students look for patterns to figure out the rule. There are many ways to play Exploding Dots. In this function-machine-like version, the facilitator never explains how it works.

I drew a row of boxes on the board. “Watch what I do on the board. When I point to you, it’s explosion time. Make explosion noises when I point to you.” I started humming a jaunty tune while putting some dots randomly spread across boxes. When I placed a fifth dot in the right-most box I pointed to the students, they said “Boom!” and I erased all five dots in that box and put a dot into the adjacent box. We kept on like this for a little bit. I did not explain to the students what was happening or how this worked. In fact, I did not say a word. Not a word. I just drew and erased dots and pointed at the students when they were supposed to make noise. Here is how it went:

1. I filled in dots.
2. The students made the indicated explosion noises.
3. I moved the dots.
4. The students made the indicated explosion noises.

I repeated this four-step process a few times then asked for the rule.

We did this a number of times in various bases until the students understood what was going on (Figure 5). I never told the students how I knew when to call for an explosion. I just moved the dots. Then I asked them to describe what was happening mathematically. This task was tough, and we ended class with students positing conjectures about the underlying structure of Exploding Dots [17].

5 Session Three

5.1 Some Math History: Navajo Code Talkers

We began class three with a lively discussion of the history and cryptology of the Navajo Code Talkers of WWII. Almost everything in this discussion was new to everyone, but Ezra helped out by explaining who Marines and anthropologists are [11].
5.2 The Signaling Problem: A New Challenge

We then returned to last week’s question of whether we had really solved the Signaling Problem. We reviewed both codes that the students invented, noticing that these codes were really codes representing another code. ⁴ As we were congratulating ourselves on devising not just one code for a code, but two, Mavis proposed that we could embed the message more deeply by alternating codes by day. So we had a code for a code for a code. “Not bad!” I thought.

I announced that the students were now ready for “level three” of this question. In level three, the challenge is more like that of the Navajo Code

⁴Remember, in the original problem, 27 meant “safety,” and the question was how to signal 27 without enemies deciphering it, since we thought the enemies may have cracked the code that 27 meant safety.
Talkers: there are many messages that must be encoded. In fact, there are 30. Could our codes handle that? The students thought for a second then shook their heads “no.”

5.3 Exploding Dots: Making Predictions

We took a break from the problem to finish class with another round of Exploding Dots. To ensure that the students figured out and could then apply the rule, within each box I filled the dots in evenly spaced rows of five as we went along, instead of the random placement we used last week. I hoped that this dot arrangement would make counting easier and make it obvious that the explosions were happening at predictable times. This time, I worked only in base ten and exploded the dots systematically from right to left, with a goal of students making the connection between Exploding Dots and their own math experiences outside of Math Circle. I asked the class what I was doing differently from last week. When everyone figured that out, we talked about that often-ignored area of mathematics: counting strategies. Then I asked them to predict explosions: “How many dots must be added to generate an explosion?” This task was essentially a subtraction exercise, which everyone was able to do.

I asked the students to predict what would happen if I placed 30 dots in the right-most box. Many conjectures sprang forth:

“Write down all the dots and move them 10 at a time,” suggested one student.

“Move them all at once and make it explode three times,” proposed Juan-Carlos.

“Just put three dots in the middle box,” countered Alex.

“Just point to us without writing all 10 down and we do the explosions,” commanded someone else.

Most people agreed that all of these strategies would work. The students wanted to dramatize the explosions. I proposed putting all 30 dots in that box. Alex expressed doubt that I could do it. So I did, giving us another chance to play with counting strategies. We ended session three on the following notes: a reminder that next week we will enter level three of our Signaling
Problem, a suggestion by Daniel that we consider Morse code, and a suggestion/demonstration by Ezra that Pig-Latin might help.

6 Session Four

6.1 Math History: More Code and More Plato

We played Exploding Dots as students gradually trickled in when Alex arrived with a colorful contraption that she had made. Everyone started asking her about it at once. She explained what Morse Code is and how her Morse Code generator worked.

Then, since our group had also been very interested in using shadow puppets as a solution to our problem, my assistant and I told them a story about shadow puppets: Plato’s Allegory of the Cave [2] (Figure 6). The students spent some time trying to make sense of this story. We then moved from sitting on the rug back to the table.

Figure 6. Plato’s allegory of the cave.
6.2 Exploding Dots Addition

I told the students to try to add two numbers: 279 and 568 [17, p.5]. The students, most but not all of whom had seen three-digit addition before, hesitated.

“Use whatever approach you’d like. I’m going to use Exploding Dots.”

“So, wait, we’re trying to add with Exploding Dots?” someone questioned.

“Any method you want,” I replied.

People worked in silence. Some grabbed pencils and started writing or drawing, while others contemplated in stillness. Several minutes later, I asked for answers. Cloe and several others hesitantly lifted their hands. Results varied: 847, 837, 747, 843, and 847. Juan-Carlos rechecked his addition and changed his answer.

I then put on the board the number 279 in boxes and the number 568 underneath that, and finally the result of 7|13|17 at the bottom. The students looked dubiously at the board. Someone said, accusingly, that my number had five digits (theirs had three).

“It’s still right though, isn’t it?” I asked.

“That would be too many numbers,” explained Alex. “My dad told me, he teaches me math, that you have to carry the one,”

“What’s carrying?” I asked. Silence.

“It’s just something that you do,” someone said finally.

Since it seemed that no one knew what carrying was, I went on playing addition with Exploding Dots with the hope that the students would make the connection. They did not. I let it go. The real focus of this course is different base systems, and that is where the students’ interests lie. I did not redirect the course into an in-depth study of addition, which I might have done had we more than six sessions scheduled. We were at the point where the initial struggle with place value had become something the students engaged in with ease. The students had finally discovered the underlying structure of Exploding Dots. So I chose not to lose class time on making connections to more
conventional methods. But we did finish the problem using Exploding Dots to get the answer into conventional notation.

“So, if only nine dots are allowed in the right box, and there are 17 in there, do we need to explode?” I asked.

“Yes! BOOM!”

That left us with the number 7|14|7. Time for another explosion: “BOOM!” One more explosion produced a final answer that was much more conventional: “847!” [14].

6.3 Exploding Dots: Convention and Precision

After this problem, we added 525 to 546 using Exploding Dots, but with only our usual three boxes on the board (Figure 7). The group debated what to do with the extra digit in the answer, and finally agreed to add another box for the thousands place value, getting 1,071 as the answer. Before agreeing on this, we had the answer 10|6|11 on the board. I asked, “Is it okay to write the number like that?”

“It’s not proper,” said the kids.

“What do you mean by proper?” I countered.
“It’s not correct,” they clarified.

“What do you mean by correct?” I asked.
“It’s not right,” they insisted.

“What do you mean by right?” I demanded.
“The opposite of left...” Daniel trailed off.
“. . .and the opposite of wrong!” the rest of the group chorused.
“It’s not good,” they explained.

I could have gone much further in my demand for precision but did not want to lose momentum on the problems at hand, so I finally I let the students off the hook. I told them that questioning everything and defining your terms are important skills in mathematical thinking. Then I explained that 1,071 is a convention we use so that everyone can communicate mathematically. I
needed to give the definition of convention, since no one was familiar with the term except for one student who had heard of beer conventions. Then I stated that the way I showed the answer on the board, with the vertical lines and seemingly extra digits, was technically valid in mathematics. It is just not the convention. “What do you mean by ‘technically’?” challenged Joanna. I was thrilled at this question that revealed the emerging skills of questioning and defining!

7 Session Five

Our penultimate class had arrived, and the students showed no signs of connecting Exploding Dots to the Signaling Problem. I suspected that they saw Exploding Dots as a pleasant diversion from the work at hand, kind of like doing a crossword puzzle as a break from work. I also suspected that they had not become fluent enough yet with different bases for the binary solution to
occur to them. So I introduced a new chapter of our problem, one that also relied on different base systems.

I posed the Alien Problem after I had begun class with a few rounds of child-directed Exploding Dots, first in decimal, then in octal. We discussed different base systems and King Charles XII of Sweden’s contribution to them [4]. We even attempted to convert from octal to decimal. A few students suggested a novel way to do this, but most did not understand this conversion. I let that go, since our Math Circle is not a race to “get it,” but instead a forum to develop mathematical thinking in a collegial environment. The mathematics that I wanted kids to own from this course is that counting strategies exist and that our number system depends upon the idea of grouping. Once kids grasp the utility of grouping, arithmetic makes intuitive sense and the generalizations that define algebraic thinking can emerge naturally. Robert Berkman [1], who created the Alien Problem, extends the problem into algebra beautifully in his article.

7.1 The Alien Problem

I asked the class the new question, integrating the question Berkman [1] posed with our scenario:

_Some alien spies have been helping our captain. It turns out that you all were right: there are enemies hiding on the field behind the house. The captain wants to know how many. The alien photographer comes down and takes a photograph of them, gives it to her captain, who writes it down and gives it to our captain. The alien captain wrote either seven, 13, or 23. Which is correct?_

This problem can perhaps be more clearly understood by an adult as this:

_What quantity is represented by a diagram of thirteen dots arranged in the shape of the Arabic numeral seven when the quantity is being communicated via diagram by an alien with two fingers on one hand and three on the other?_

Younger students, however, would likely have a less frustrating experience understanding it via the story and illustration. Also, it can be good Math Circle pedagogy to ask a question in a vague way that requires questioning to arrive at precision.

I drew a diagram of the “photo:” a square containing 13 dots, each representing an enemy, in the shape of the number seven (Figure 8). The kids
debated whether seven or 13 was correct until Ezra questioned, “Why would 23 even be on this list?” This turns out to be the real meat of the problem. Ezra’s question produced many very insightful conjectures about groupings of numbers. I then drew a diagram of an alien with two fingers on one hand and three on the other, leading to further conjectures, but none were fruitful in the long run.

![Diagram of Alien with Numbers]

Figure 8. Collaborative exploration of the Alien Problem.

In our exploration of the Alien Problem, I revisited the term decimal when the conjectures came to a standstill. I asked the students, “What does decimal mean? What are digits? What does the word digit mean in Latin? Why do we count the way we do? Why is it easy for us to use base ten (a.k.a. decimal)?”

When the group discussed these questions and practiced counting on their fingers, the Alien Problem started to make sense. The students figure out that 23 is the base five expression of the decimal number 13. The students instructed me to add more choices to the initial numerical-only choices: all of the above, none of the above, and one or two of the above. Just as we ran out
of time, the students arrived at consensus: their answer was one or two of the above.

They left asking to act out the Signaling Problem next week in our final session. I promised that after we finished level three of the Signaling Problem, namely devising a code that can send 30 messages, we would do so.

8 Session Six

8.1 Exploding Dots: Binary, Base One, and Technology

We began this final session with a few rounds of Exploding Dots, this time in binary (Figure 9). The students had great fun vocalizing the large number of explosions in binary compared to decimal:

BOOM! BOOM! BOOM! BOOM! BOOM! BOOM! BOOM! BOOM!

Figure 9. Five-box binary Exploding Dots machine. (Right photo by A. Hoppel.)

I asked how many different numbers could be represented in binary with just five boxes for dots (units place through sixteens place, as calculated by the kids). The kids were unsure about the math but played with maximizing the number of explosions. Cloe proposed a rule that once the sixteens box was “full” (ready to be exploded), we put the appropriate number of dots in the units box, creating a continuous repeating loop and therefore nearly continuous explosions:
Juan-Carlos, Daniel, Alex, and Joanna loved this idea. Ezra and Mavis were more interested in the mathematics than the explosions.

Alex extended Cloe’s idea with a proposal that we use base one to generate non-stop explosions. We discussed whether base one could even exist, concluding that it cannot. I was reminded here that following kids’ joy and curiosity led to some deep mathematics, such as why base one cannot exist. We then returned to binary. It took some hard work, but most of the kids did conclude that we could represent 31 different values in the five-box binary machine, since 32 would require an additional box.

In the previous week the kids had written numbers on slips of paper for use in the Exploding Dot machine. This week we did a few more of these, with the students deciding which base system to use for each number. I was certain that they would choose the ease of decimal for the large numbers over the tedium of binary. I was wrong. They wanted more explosions, so chose binary. We discussed how different systems are useful in different scenarios.

I then began a discussion of the use of binary in computers and robots. According to Robert Kaplan, “explaining to them after that their TV, phones, etc. all work in binary because of the simplicity of on/off, is usually a huge revelation” [personal communication, October 31, 2012]. My students, however, were more interested in acting out the Signaling Problem than in discussing how zeroes and ones can represent on and off switches. Maybe this was a matter of different students, different results. Or perhaps my discussion was not that compelling; I am not that interested in technology so the students may have picked up on my faked enthusiasm. This is why I generally only select Math Circle topics that are exciting to me. Everything in this course was exciting to me except the connection to technology. Perhaps, however, the students’ disinterest was just bad timing; how can anything beat planning a show, especially one with math in it?

8.2 The Signaling Problem: Nudging Hard (Too Hard?) Toward a Solution

I left the five-box binary Exploding Dots machine on one board (Figure 9) and moved the group to the board on the other side of the room where we were working the Signaling Problem. It was a big room, so we all had to walk over. There was some physical distance between Exploding Dots and the Signaling Problem.
We reviewed the two codes that our group had devised in prior weeks to solve the lower levels of the Signaling Problem. All were still stumped, however, on how to solve level three, in which 30 different messages needed to be sent. A solution to this level of the problem was on the other board, in Exploding Dots format. None of the kids made the connection. I kept thinking, “It’s right there! It’s RIGHT THERE!” I tried to project that thought onto the students without saying it or pointing or jumping up and down. We were running out of time. I questioned and nudged, nudged and questioned. More engineering approaches emerged, but no more mathematical approaches sprang forth.

Since this was the final session, I finally bit the bullet and reframed the question: “Could you use binary to solve this problem?” “Yes,” said the students, at first. Then, after a bit of thought, the group consensus changed to “No.” No one believed that one candle turned on or off in each of five windows could make up as many as 30 combinations. These children were unaware of the power of combinatorics. I pointed to the other board, containing the work we had been doing earlier in the session. I asked, “How many numbers can you represent in a five-box binary Exploding Dot machine?” “31,” replied everyone. “How many messages can you represent in a five-window system with two values per window?” I asked. No one was sure, but all reiterated that is certainly was not as many as 30.

All of my nuanced approaches to help the students discover a solution for themselves were coming to naught. If the students were going to solve the problem during the course, I needed something stronger immediately.

8.3 Using Counterexamples

I asked them to name a code number that would not work using candles to express the number in binary. Daniel suggested 86. Joanna suggested 200. Juan-Carlos suggested trying numbers we already were using in the code: 13 and 36. We found that 13 worked, but the others did not. I said nothing.

Juan-Carlos then proposed that we change the code so that only numbers that do work with this system be used. “What numbers would those be, then?” I asked. The kids now suspected that any number from one to 31 would work and tried some (Figure 10). We did not try all these numbers because the group grudgingly was able to see the connection to Exploding Dots and to generalize that all of these numbers would work. I say grudgingly because (1) this solution seemed to defy their number sense, (2) they wanted to try more numbers, and moreover (3) they wanted to get on with performing the play.
So, binary did work with the candles, but what really excites me about this solution is how the kids came to it. No one saw the solution until we made the assumption that a proposed solution did not work, and I charged the group with proving that it did not work. The kids then were able to determine that certain examples did in fact work. This is an important type of mathematical reasoning, and children can do it!

Figure 10. Students’ final answer to level three of The Signaling Problem.

8.4 End of the Course

I turned off the lights. The students got out their flashlights and performed. Juan-Carlos, Ezra, Mavis, Cloe, and Alex played people in the windows; Joanna played the captain; and Daniel, the alien. Ruby and Matteo were absent this day. The students acted out all three codes that emerged over our six weeks. Unsurprisingly (since the Kaplans had mentioned this to me) the proper right/left orientation in binary was confusing [Robert Kaplan, personal communication, October 31, 2012]. It was a short play, and a fun play. Then everyone went home. In my final weekly report to the parents, I wrote:
I hope that the kids have gone home with a stronger sense of place value, digits, and different number systems. They might not know these terms but do understand these concepts. To follow up on these concepts at home, I would recommend using Exploding Dots to do more addition, and to do subtraction, which we never had time for. The Dots make “borrowing” and “carrying” intuitive. I would recommend that the older kids in the group try more alien arithmetic as a way to move into algebraic thinking. Everyone would probably enjoy more discussion about Plato and uses for different bases. I like to think that everyone will continue to jump out of their seats with excitement as their mathematical thinking evolves.

9 Reflection on Implementation

Did I meet the goals that were stated in the intro/overview? In the spirit of the pedagogical principle “ask, don’t tell,” I invite you to tally up your yesses versus nos. Did I avoid lecturing? Did I avoid textbooks, grades, tests, and homework? Was there collegial conversation? Did I allow and encourage questions, challenges, and mistakes? Did I avoid competition? Was the group small? Was the problem accessible? Was the problem mysterious? Did it lead “behind appearances to the underlying structure?”[8] Did students use a lot of “imagination, goodwill, and conversation with each other to uncover” that underlying structure? [ibid]. Did the students still have doubts at the end? If so, were those doubts informed doubts? Did the students discover a solution instead of being told it? Did the students own it? Did I learn from the students? Was it relevant? Was it joyful? Was it uplifting?

9.1 My Evaluation

Thanks to these great problems designed by Kaplan, Berkman, and Tanton, I think that I achieved most, but not all, of these pedagogical goals. To me, and possibly to you, most of the answers are “yes” and self-evident except for two. I missed a big opportunity to connect the Signaling Problem to what the students were learning in the classroom. The connection between “carrying” and Exploding Dots was so relevant to the mathematics the students were learning elsewhere; I wish I had facilitated its discovery. Maybe I could have spent less time on math history to achieve this. More importantly, I regret how much I nudged the students toward a solution in the last session. Again, had I not spent so much time on math history, maybe we would have had more time for discovery. Another option would
have been to leave the problem unsolved and allow the students to explore it on their own. Or I could have scheduled another session to work on it more. If I do this problem again, I will try to make this happen.

Some of my “yes” answers warrant some comment here. Student collaboration was so prevalent and necessary in this course. The group solution to the lower-level Signaling Problem was only possible from every student contributing conjectures and questions. On the first day of class, had I told these students who really struggled with adding three-digit numbers in base ten that they would be solving problems in binary, they may have become discouraged before even hearing the accessible mystery.

I must mention here that Exploding Dots does not really need an accessible mystery; it is compelling on its own. I coupled it with an accessible mystery because my deep mathematical goal was for students to discover for themselves that the underlying structure of something might be revealed to be binary. I wanted to open their minds to the fact that decimal is not the only number system, and that other systems sometimes make more sense. The students really did end up owning their mathematics. They wrote part of the problem (remember the princess?) and several times did my favorite thing to happen in Math Circles: ran up to the board, grabbed the chalk, and took over. In fact, sometimes when that happens I leave the room and pretend to need to excuse myself for an “emergency phone call” while I observe from around the corner. I did learn from the students.

Coming in to this course, I knew nothing about shadow puppets or Morse code or Plato. I learned from the students about the first two and was motivated by them to learn about the third. Also, it is worth noting that I knew nothing about Navajo Code Talkers. I only knew that they existed. But because I wanted to tie the abstract mathematics to the concrete real world, I found joy in researching this topic and sharing what I learned with the class. And the students found joy in the course. There was much laughter, standing up, and everyone talking at once. It seemed that the students never wanted to leave. Again, achieving these pedagogical goals was so much more feasible because of the problems I chose. Does your evaluation match mine? We do not have to agree. It is fine to have doubts, so long as they are informed doubts.

9.2 Lessons Learned and Pedagogical Suggestions

I learned that using the Signaling Problem facilitated a course in which students discovered underlying mathematical structure in a joyful way. Choosing an accessible mystery that others had previously used successfully made my job much easier.
I also learned that it is possible to witness the development of students’ mathematical confidence in a short period of time. In session one, they accepted what I said as truth. Very quickly, they were arguing and collaborating with each other. By session four, they were arguing against my conjectures. By session five, I was following the collective intellectual drive of the class.

As the students became discouraged, a natural part of the mathematical process, I learned that there are ways to overcome discouragement that can be coached. When the students became discouraged as the difficulty of the problem ramped up, they got re-engaged by (1) taking some math-history interludes to rest their brains from this difficult problem, (2) harnessing their questions and conjectures into lists, and (3) going back and forth between two seemingly unrelated mathematical explorations. They struggled with the tug between the opposing desires: the desire to poke holes in every conjecture; and the desire to say enough is enough, let us accept it without proof. I found myself making both macro and micro adjustments throughout the course to keep students engaged when the going got tough and to sustain the engagement when the going got good.

The students did not solve the problem as soon as I thought that they would. Because we only had a limited amount of time, I reacted by introducing the Alien Problem to facilitate the solving of the Signaling Problem. I witnessed the shift from the students reacting to me in the first half of the course to me reacting to the students in the second half of the course. I therefore learned to let go of my own expectations of how the course would unfold.

The following list offers some explicit guidance for Math Circle leaders who seek to implement this style, especially if you are new to working with students this young:

· Prepare in advance some comments to say in response to students’ conjectures (or lack of conjectures), comments that can engender deeper student thinking. The list “Becoming Invisible” [9] is a great source for such comments.

· Do not be afraid to say, “I do not know.”

· Keep running lists of students’ questions, conjectures, and assumptions on the board.

· Have other activities available for when the students become too frustrated or restless, for instance function machines, historical anecdotes, physical math activities, activities that use manipulatives, and books
with logic problems. I keep three things in my bag at all times: sidewalk chalk, colored wooden cubes, and a book of logic puzzles.\textsuperscript{4}

- Have the students sit at a table or in a circle or on the floor with you, and not in rows of desks.
- Keep your group small or split a larger class into smaller groups.
- Keep the age range narrow if possible. The younger the students, the narrower the range.
- Let go of attachments to a strict agenda.
- Choose topics that are interesting to you.

During these six weeks, I learned that it is possible to see first-hand how mathematical thinking emerges. The students spent a lot of time just understanding the problem. They did not need any coaching on forming and rejecting conjectures, but they had to learn how to make and state assumptions. I also saw the students’ mathematical competence grow. They grappled with place value because they had been doing things by rote without a full understanding of underlying structure. Making predictions in Exploding Dots gave them the opportunity to invent their own mathematics. By the end of the course, the students developed a firm understanding of how place value works. They could comfortably work in number systems other than decimal and learned that other number systems sometimes make more sense. The students’ curiosity also led me into some unexpected directions, including base one, maximizing explosions, Plato, and Morse code.

Acknowledgements

I thank Rachel Steinig for assisting in my research by writing a blog report about part of session four in the course described in this article and for taking notes and capturing student quotes during all six sessions. Rachel also assisted in leading the sessions whenever extra facilitation was needed. I thank Robert Kaplan (The Global Math Circle) and Ellen Kaplan for providing the Signaling Problem and for providing guidance on how to implement it. I also thank Talking Stick Learning Center for institutional support, the Mathematical Sciences Research Institute for financial support of the Talking Stick Math Circle, and Angela Hoppel and Asha Larsen for photography.

\textsuperscript{4}I like [5] for older elementary students, [6] for younger elementary students, or [13] (especially chapters one through three and six) for everyone.
References


