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A Gentle Introduction to Inequalities: A Casebook from the Fullerton Mathematical Circle

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Cover Page Footnote

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Run for nearly a decade, the Fullerton Mathematical Circle at California State University, Fullerton prepares middle and high school students for mathematical research by exposing them to difficult problems whose solutions require only age-appropriate techniques and background. This work highlights one of the avenues of study, namely inequalities. We cover Engel's lemma, the Cauchy–Schwarz inequality, and the AM-GM inequality, as well as providing a wealth of problems where these results can be applied. Full solutions or hints, several written by Math Circle students, are given for all of the problems, as well as some commentary on how or when to assist students, and details about the pedagogical value of certain problems.

Keywords: AM-GM inequality, Cauchy–Schwarz inequality, Engel's lemma, problem-solving

1 Introduction

Our goal in the Fullerton Mathematical Circle (FMC), an outreach program of the Department of Mathematics at California State University, Fullerton, is to introduce middle and high school students to the problem-solving culture as early as possible. We have found that the study of algebraic inequalities provides a perfect point of access to math competitions, mathematical problem-solving, rigor, and an appreciation for the power and beauty of mathematics. There are a wealth of techniques and strategies around inequalities which have

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no Calculus prerequisite—only algebraic methods, and some cunning—which makes this perfect material for outreach to younger students.

As early as 1903, the Romanian Mathematical Society's monthly periodical *Gazeta matematică* started publishing original problems designed for middle-school students. This was how Gheorghe Ţiţeica (see [2]) and his colleagues from the editorial board promoted mathematical creativity from an early age. (For a celebrated product of that school in the first decades of the 20th century, see [6]). They understood how important it is to find perpexling, interesting, age-appropriate questions to inspire and attract young minds towards mathematics. Arguably, this is the key quest in all of mathematics: *What are the good problems?*

A central element in our program at the FMC since its inception in 2011 has been to translate to English the problems in the *Gazeta* and to propose them to our students. Sometimes, students arrive at solutions almost immediately, and then our task is to encourage them to explain their thinking to others so that other students gain inspiration and insight. Sometimes, all attending students struggle with the problems, and we spend more time in the presentation of solutions and cultivating problem-solving techniques. Over time, we have witnessed these young mathematicians produce a litany of original solutions to these problems, many of which have been submitted to the *Gazeta*, or which formed the foundations of projects that culminated in student poster presentations or even full conference talks!

The methods developed in this curriculum—applying Engel's lemma, the Cauchy–Schwarz inequality, and the AM-GM inequality—are elementary, but they can be applied in very clever ways involving forethought and sneaky substitutions. The bridge from one to the other is experience, so offering opportunities for students to see and practice these techniques every week helps excite new students and helps craft the exceptionally talented and dedicated students into great mathematical thinkers. The repeated use of a core set of methods also helps obviate the need for a linear progression of ideas. Such a progression is incredibly challenging, if not impossible, to maintain in the face of the sometimes sporadic attendance one would expect from students who have weekend sporting events, family obligations, or just the need to take a Saturday morning off.

The material included here is principally intended for fairly advanced high school students, though we have had some exceptional middle school students thrive in our FMC. We draw students from a nearby STEM magnet school, and particularly eager students commute from neighboring counties. While the curriculum here may not be applicable for all Math Circles, we hope that it will be a useful guide and resource for anyone with motivated and enthusiastic learners. Students regularly surprise and impress us, creating an empowering experience for them and for us. The material, particularly terminology and notation, may need adaptation for younger or lesser-prepared audiences. Even our presentations are highly interactive and adaptive to the audience we have on any given Saturday, and intended lessons often detour considerably. However, regardless of ability, all students benefit greatly from being challenged and encouraged to hone their skills. And the reader need not be frightened by references to Mathematical Olympiads or the Putnam Exam. It is a testament to the elegant power of such elementary methods that selected problems, even from notoriously difficult exams, may yield to these tools.

In Section 2 we lay out the general pedagogical approach of our Math Circle. In Section 3 and Section 4, we provide two samples of class content, each consisting of an introduction to the main tool (the Cauchy–Schwarz inequality and the AM-GM inequality, respectively); an array of problems on which students can apply their new knowledge and skills; and an integrated discussion of some common pitfalls and highlights. In Section 5, we provide a couple of additional problems for advanced students or regular attendees.

There are many resources for more problems and more general theory. Perhaps the most influential book on the topic of inequalities is the classical volume [9], while a more accessible reference is [19]. In our presentation, we have used several arguments and proofs from a wonderful little booklet [13] of Korovkin coming from the Soviet classical school. We also present some of the solutions found by our students to selected problems. The goal of this work is certainly not to be comprehensive, but to serve as an introductory resource, and as a springboard for other Math Circles interested in attempting our approach.

2 Pedagogy of the Fullerton Mathematical Circle

How can problem solving be taught? Is there any educational strategy in preparing students for approaching new and difficult mathematical problems? The answer definitely depends on one's audience, and at the Fullerton Mathematical Circle we have been particularly fortuitous to have encountered a superb community of students with many interesting and creative ideas. More importantly, connecting with that community helps us recognize their potential. Our guiding principle is to treat every Math Circle participant with the attention and individual guidance usually granted to graduate students.

First and foremost, we teach our students content, and then we guide them towards problem-solving. The key premise is that once the technical ideas are taught, the students can reach solutions by themselves. Our sessions typically last two hours, with the first fifty minutes devoted to direct instruction, usually in the form of lectures or guided explorations, and the second fifty minute block is devoted to the students actively solving problems. Each block is followed by time to socialize and light refreshements.

Though we have many students who will attend the sessions regularly, there are many who will only show up periodically. This makes it challenging to consistently introduce new material during the weekly lectures. Instead, we tend to cover a few topics repeatedly, but from many different angles, and with many different toy examples. This allows for the regular attendees to gain a much deeper understanding of the topics, while keeping the level of any individual session adequately accessible.

An expectation our students have is that they will not be allowed to be passive learners. The second half of the session lets the students participate in the fun of problem-solving. We prepare a handout, including many problems from the *Gazeta Matematică* and other sources, and give this to the students. The problems are broken up by difficulty level and we typically arrange the students in groups according to their mathematical background and oral communication abilities.

Generally, it is better to have students working together when they are closer in mathematical background, experience, and aptitude. However, it is also very useful to keep in mind how well a student communicates with their peers. For example, while it may often be reasonable to place an advanced student with someone less experienced, if the advanced student struggles to explain their ideas, then there is a good chance that the less experienced student will be left out of any collaboration.

Frequently, our students will get stuck on problems—we shy away from using too many problems that the students will find straightforward. It is extremely important to identify what type of obstruction is blocking their path. We tend to spot three different classes of obstructions.

Statements

This is where students, usually new to the program or topic, do not yet have the definition of a concept or the statement of a result internalized. Frequently, this will take the form of not knowing the hypotheses they need to check in order to apply a result. In this case, we usually ask them to review the statement, or encourage one of their groupmates to share their understanding of the statement.

Routine Applications

Given a formula, some students may struggle in determining what ingredients given in the problem are relevant, or how they fit into the formula.

This is where having some extra straightforward problems prepared, but not necessarily included on the worksheet, can be beneficial. Giving students problems where the application is more transparent usually succeeds in allowing them to clear this hurdle, oftentimes before they even finish going through the supplementary problem.

Clever Applications

Even our seasoned verterans will get stuck on problems where using their toolbox requires a non-standard substitution or transformation of the problem. Unsurprisingly, this is where we see our students most frequently perplexed. This is where we give the most help, but also where we let them struggle the most. The outcome is some of the most creative mathematics we get to see.

We firmly believe that learning to communicate is an absolutely necessary skill for our students. So, after students have solved problems, we invite them to present their solutions to the class. We make it a point to get as many different students presenting at each session, and try to foster a collaborative atmosphere where comments from the audience are considered constructive rather than critical.

In the following sections, we give a taste of the material we cover, as well as a sampling of the problems we solve. Moreover, we give examples of steps we expect students to find difficult, as well as some of the hints we provide in these situations.

3 The Cauchy–Schwarz inequality

An incredibly important and ubiquitously used inequality is the Cauchy–Schwarz inequality. In this section, we provide a guided exploration to prove the classical Cauchy–Schwarz inequality via another useful result, Engel's lemma. We then provide several example problems for students to attack using this tool.

3.1 A guided exploration

Theorem 1 (The Cauchy–Schwarz inequality)**.** *If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real numbers, then*

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \cdot \left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2 .$$

Equality holds when there exists a real number c such that $a_1 = cb_1, a_2 = cb_2, \ldots, a_n = cb_n$.

An oft-cited proof (e.g.[10]) considers the quadratic function

$$f(x) = \sum_{k=1}^{n} (a_k x + b_k)^2,$$

using the observation that the discriminant must be non-positive. While this proof is accessible to high-school students, who will have encountered properties of parabolas, the following development is considerably gentler, and hints at how powerful it can be to simply apply known results about inequalities to obtain new insights.

We let students work to prove each of the following lemmas before presenting the proofs.

Lemma 1. *If $x, y > 0$ and a and b are any real numbers, then*

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$$

with equality occuring if and only if $\frac{a}{x} = \frac{b}{y}$.

Proof. Since x and y are strictly positive real numbers, we can multiply the relation by $xy(x+y)$ to obtain the equivalent inequality

$$a^2 y(x+y) + b^2 x(x+y) \geq xy\left(a^2 + 2ab + b^2\right).$$

After the appropriate distribution and cancellation, this relation reduces to

$$a^2 y^2 + b^2 x^2 \geq 2abxy,$$

which is readily recognizable as

$$(ay - bx)^2 \geq 0.$$

This inequality holds for all x, y, a, b, with equality occuring precisely when $ay = bx$, or equivalently $\frac{a}{x} = \frac{b}{y}$.

Commentary. Most of our students have not taken calculus, so the notion of maximum and minimum values is often a foreign concept. This provides an opportunity to establish a foundational idea that will be seen again. Also, students must become very aware of the hypotheses in a statement, particularly those regarding the signs of the variables. Furthermore, they must become adept at inequality multiplication manipulation, namely, knowing whether an inequality will reverse directions when they multiply by a variable expression. We provide direct instruction on these points, but we expect students to struggle for a while when they start, and adjust the time and attention spent explaining details as appropriate.

□

We pursue our exploration further with the following claim.

Lemma 2. *If* $x, y, z > 0$ *and* $a, b, c \in \mathbb{R}$*, then*

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

Equality holds precisely when there is a constant k *such that* $a = kx, b = ky$*, and* $c = kz$*.*

Proof. Indeed, by applying the previous result twice, we see that

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b)^2}{x+y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

To verify the equality claim, Lemma 1 implies that $ay = bx$ and that $(a+b)z = c(x+y)$. If we instead first combine the second two terms, then, in the equality case, observe that we have $bz = cy$. With the previous equality, we conclude that $az = cx$, or $\frac{a}{x} = \frac{c}{z}$. As $a = (a/x)x, b = (b/y)y$ and $c = (c/z)z$, the equality case in Lemma 1 shows that $k = a/x = b/y = c/z$ satisfies $a = kx, b = ky$, and $c = kz$. □

We are ready now to state Engel's lemma, the logical conclusion of the previous explorations.

Lemma 3 (Engel's lemma)**.** *If* $x_1, x_2, \ldots, x_n > 0$*, and* $a_1, a_2, \ldots, a_n \in \mathbb{R}$*, then*

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + \cdots + a_n)^2}{x_1 + \cdots + x_n}.$$

Furthermore, equality holds precisely when there is a constant k *such that* $a_i = kx_i$ *for* $i = 1, \ldots, n$*.*

A proof can be given immediately by iterating $(n-1)$ steps as in the explorations above. Moreover, Engel's lemma implies the Cauchy–Schwarz inequality! (Depending on the strength of the students, this could also be a time during the lecture to introduce proofs by induction, though we find that usually this takes us too far off-topic.) Again, we let students struggle with this proof before presenting it.

Proof of The Cauchy–Schwarz Inequality. If any of the numbers b_i were zero, we could simply ignore them. So, without loss of generality (a fantastic term for students to encounter early!), we may assume that all the real numbers $b_1, b_2, \ldots, b_n$ are nonzero. Engel's lemma implies that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = \frac{a_1^2 \cdot b_1^2}{b_1^2} + \frac{a_2^2 \cdot b_2^2}{b_2^2} + \cdots + \frac{a_n^2 \cdot b_n^2}{b_n^2} \geq \frac{(a_1 b_1 + \cdots + a_n b_n)^2}{b_1^2 + \cdots + b_n^2}.$$

Since $b_1^2 + \cdots + b_n^2 > 0$, cross-multiplying we obtain the Cauchy–Schwarz inequality.

> *Commentary.* Sometimes students will arrive at this proof, and other times will be at a loss. Either way, we generate a group conversation about how to transform what we want, one side of the Cauchy–Schwarz inequality, to resemble something we know, one side of the inequality from Engel's lemma, and how powerful is the innocuous idea of multiplying by a clever form of 1.

□

3.2 Some problems

We next explore several direct applications of Engel's lemma.

Despite the fact that this next result is well-known, we introduce it in Math Circle as a "problem," as opposed to a theorem, because students are able to work it out themselves after the previous discussion! Some students will connect this problem to Engel's lemma, and others to the Cauchy–Schwarz inequality, so we present both lines of reasoning elicited from students. Only after they have discovered a solution (or both solutions) do we reveal that they have reproved a classical result on their own.

Theorem 2 (Nesbitt's inequality)**.** *If $a, b, c > 0$, then*

$$\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} \geq \frac{3}{2}$$

with equality holding if and only if $a = b = c$.

Proof via Engel's lemma. Multiplying the summands on the left-hand side by c/c, a/a, and b/b, respectively, and applying Engel's lemma, yields

$$\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} = \frac{c^2}{ac+bc} + \frac{a^2}{ba+ca} + \frac{b^2}{cb+ab} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

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This is greater than or equal to $3/2$ if and only if

$$(a+b+c)^2 \geq 3(ab+bc+ca),$$

that is

$$a^2+b^2+c^2-ab-bc-ca \geq 0.$$

By doubling both sides and factoring, we get the equivalent inequality

$$(a-b)^2+(b-c)^2+(c-a)^2 \geq 0,$$

which holds for all a, b, c. Furthermore, equality occurs precisely when $a = b = c$. □

Proof via the Cauchy–Schwarz inequality. In our attempt to morph the $n = 3$ version of the Cauchy–Schwarz inequality:

$$\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right) \geq \left(a_1b_1+a_2b_2+a_3b_3\right)^2$$

into Nesbitt's inequality, we make the following substitutions:

$$\begin{aligned} a_1 &= \sqrt{b+c}, \quad & a_2 &= \sqrt{c+a}, \quad & a_3 &= \sqrt{a+b}, \\ b_1 &= \frac{1}{\sqrt{b+c}}, \quad & b_2 &= \frac{1}{\sqrt{c+a}}, \quad & b_3 &= \frac{1}{\sqrt{a+b}}. \end{aligned}$$

Plugging these values into the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} 9 = (1+1+1)^2 &\leq \Big((b+c)+(c+a)+(a+b)\Big)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \\ &= 2(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \\ &= 2\left(\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\right) \\ &= 2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+6, \end{aligned}$$

and the result now follows easily. □

In fact, we can explore considerably more advanced problems that demonstrate how much progress the students can make by simply applying or mirroring the above methods. A university student who has completed multivariable calculus, for instance, might approach the following problem using Lagrange multipliers. However, the solution here is readily accessible to any avid student with solid algebra skills.

Problem 1. *([15]) Let $x, y, z > 0$ with the property that $x^2 + y^2 + z^2 = 3$. Prove that*

$$\frac{x^3}{y^2+z^2} + \frac{y^3}{z^2+x^2} + \frac{z^3}{x^2+y^2} \geq \frac{3}{2}.$$

Solution. By Engel's lemma:

$$\begin{aligned}
\frac{x^3}{y^2+z^2} + \frac{y^3}{z^2+x^2} + \frac{z^3}{x^2+y^2} &= \frac{(x^2)^2}{xy^2+xz^2} + \frac{(y^2)^2}{yz^2+yx^2} + \frac{(z^2)^2}{zx^2+zy^2} \\
&\geq \frac{(x^2+y^2+z^2)^2}{xy^2+xz^2+yx^2+yz^2+zx^2+zy^2} \\
&= \frac{9}{xy^2+xz^2+yx^2+yz^2+zx^2+zy^2} \\
&= \frac{9}{x(y^2+z^2)+y(x^2+z^2)+z(x^2+y^2)} \\
&= \frac{9}{x(3-x^2)+y(3-y^2)+z(3-z^2)}.
\end{aligned}$$

> *Commentary.* The only real trick here is forcing the cubic terms into squared terms, amenable to Engel's lemma. This step is almost always discovered and discussed within each group.

In order to complete the proof we need only to show that

$$6 = \frac{2 \cdot 9}{3} \geq x\left(3-x^2\right) + y\left(3-y^2\right) + z\left(3-z^2\right). \tag{1}$$

We claim, for $x > 0$, that $x(3-x^2) \leq 2$. Indeed, this is equivalent to

$$x^3 - 3x + 2 \geq 0$$

or, after factoring,

$$(x-1)^2(x+2) \geq 0.$$

which is certainly true for $x \geq -2$. and hence for $x > 0$. Similar claims are true for y and z, respectively. This verifies (1), completing the proof. The last part of the proof also shows that equality is obtained only when $x = y = z = 1$.

> *Commentary.* Here is where fundamental skills may need to be reiterated or strengthened, but students are motivated by having already seen the big idea of the problem. Now their standard curriculum is put into a context of solving competitive math problems.

□

The following problem exercises precisely the kinds of technique that could be useful in a high-school student's toolbox while preparing for the *USA Mathematical Olympiad*, whose format and structure is closely aligned to the *International Mathematical Olympiad*. However, the Engel's lemma-like approach is completely tractable to middle school students trained in thinking strategically about inequalities.

Problem 2. *([14]) If $a, b, c > 0$, then*

$$a^3 + b^3 + c^3 + a^2c + b^2a + c^2b \geq \frac{(a+b+c)^3}{3 + \frac{c}{a+b} + \frac{b}{c+a} + \frac{a}{b+c}}.$$

Solution. As in the previous example, we manipulate the left-hand side so that we can apply Engel's lemma:

$$\begin{aligned} a^3 + b^3 + c^3 + a^2c + b^2a + c^2b &= a^2(a+c) + b^2(b+a) + c^2(c+b) \\ &= \frac{a^2}{\frac{1}{a+c}} + \frac{b^2}{\frac{1}{b+a}} + \frac{c^2}{\frac{1}{c+b}} \\ &\geq \frac{(a+b+c)^2}{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}} \\ &= \frac{(a+b+c)^3}{(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)} \\ &= \frac{(a+b+c)^3}{3 + \frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a}}. \end{aligned}$$

□

Note. The superb problem above was written by the Bucharest-based author Alexandru Mihalcu when he was in his 9th grade!

To emphasize the importance of these techniques, it is useful to point out that, historically, the algebraic techniques to determine extreme values were discovered before the discovery of Calculus. This classical example, which can be found in [8], illustrates the point quite nicely.

Problem 3. *For $a, b, c > 0$, determine the minimal value of the expression*

$$\frac{3a}{b+c} + \frac{4b}{c+a} + \frac{5c}{a+b}.$$

Solution. An initial instinct may be to seek a common denominator for the fractions. This proves unfruitful, however. There is insight in observing that adding (a multiple of) each summand's denominator into the numerator gives a common factor.

$$\begin{aligned}\frac{3a+(3b+3c)}{b+c}+\frac{4b+(4c+4a)}{c+a}+\frac{5c+(5a+5b)}{a+b} &= \frac{3(a+b+c)}{b+c}+\frac{4(a+b+c)}{c+a}+\frac{5(a+b+c)}{a+b}\\ &= \Big(a+b+c\Big)\Big(\frac{3}{b+c}+\frac{4}{c+a}+\frac{5}{a+b}\Big).\end{aligned}$$

All we have done is to add 12, so we can rewrite our initial expression with a factorization almost amenable to the Cauchy–Schwarz inequality. Finally, the only difference between the first factor, $(a+b+c)$, and what we would like in order to apply the Cauchy–Schwarz inequality, $(b+c)+(c+a)+(a+b)$, is a factor of 2, since each variable shows up twice in the latter sum.

$$\begin{aligned}\frac{3a}{b+c}+\frac{4b}{c+a}+\frac{5c}{a+b} &= \Big(a+b+c\Big)\Big(\frac{3}{b+c}+\frac{4}{c+a}+\frac{5}{a+b}\Big)-12\\ &= \frac{1}{2}\Big((b+c)+(c+a)+(a+b)\Big)\Big(\frac{3}{b+c}+\frac{4}{c+a}+\frac{5}{a+b}\Big)-12\\ &\geq \frac{1}{2}\Big(\sqrt{3}+\sqrt{4}+\sqrt{5}\Big)^2-12.\end{aligned}$$

The minimum value, $\frac{1}{2}\Big(\sqrt{3}+\sqrt{4}+\sqrt{5}\Big)^2-12$, is attained when there is a k so that

$$\sqrt{b+c}=\frac{k\sqrt{3}}{\sqrt{b+c}},\qquad \sqrt{c+a}=\frac{k\sqrt{4}}{\sqrt{c+a}},\qquad \sqrt{a+b}=\frac{k\sqrt{5}}{\sqrt{a+b}},$$

that is, when

$$\frac{b+c}{\sqrt{3}}=\frac{c+a}{\sqrt{4}}=\frac{a+b}{\sqrt{5}}.$$

Commentary. Although we work here with particular numbers, the argument we present might be viewed as a general technique. The real art is to pick the right starting point, and to work our way into using the Cauchy–Schwarz inequality.

□

The following problem was presented for students to think about and, within 15 minutes, an 8th grader, Alvin Kim, who had been attending the FMC for a couple of years, offered the solution shown here for the class! In a Math Circle, the general progression of content cannot be as linear as in a standard classroom, but having a large story to tell and helping students incrementally advance through the arc leads to significant results. In fact, this student presented a poster at the Southern California-Nevada sectional MAA Meeting in Spring 2014 with this solution and several others borne out of his work at the FMC!

Problem 4. *([11]) Prove that, for all real numbers a, b, c, we have:*

$$\sqrt{(a+1)^2+(b+1)^2}+\sqrt{(b+1)^2+(c+1)^2}+\sqrt{(c+1)^2+(a+1)^2} \geq \sqrt{2}(a+b+c+3).$$

Solution. (*Alvin Kim*) We use the substitutions: $a+1=x$, $b+1=y$, $c+1=z$. The inequality we have to prove turns into

$$\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2} \geq \sqrt{2}(x+y+z).$$

Applying the Cauchy–Schwarz inequality cleverly to the summand $\sqrt{x^2+y^2}$ and then taking the square root yields

$$\sqrt{x^2+y^2}\cdot\sqrt{1^2+1^2} \geq 1\cdot x+1\cdot y,$$

or, equivalently,

$$\sqrt{x^2+y^2} \geq \frac{1}{\sqrt{2}}(x+y). \tag{2}$$

Cycling x, y, and z yields two other similar relations corresponding to the other two summands. Adding these three inequalities we obtain

$$\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2} \geq \frac{1}{\sqrt{2}}(x+y+z+x+y+z)=\sqrt{2}(x+y+z),$$

as desired. The equality case holds when $x=y=z$ which, in terms of the original variables, means $a=b=c$. □

Before presenting our final problem of the section, a gem from the 2015 Romanian Olympiad, we call attention to a porism of the above solution, namely that in using the Cauchy–Schwarz inequality to obtain (2), had we not taken the square root, we would have found that

$$(x+y)^2 \leq 2(x^2+y^2). \tag{3}$$

This observation is offered as a hint for the next problem, and can also be argued geometrically as seen in Figure 1.

Problem 5. *([17]) Let $a, b, c > 0$ be three real numbers such that $a+b+c=3$. Prove that*

$$a\sqrt{\frac{b^2+c^2}{2}}+b\sqrt{\frac{c^2+a^2}{2}}+c\sqrt{\frac{a^2+b^2}{2}} \leq 2(ab+bc+ca)-3abc.$$

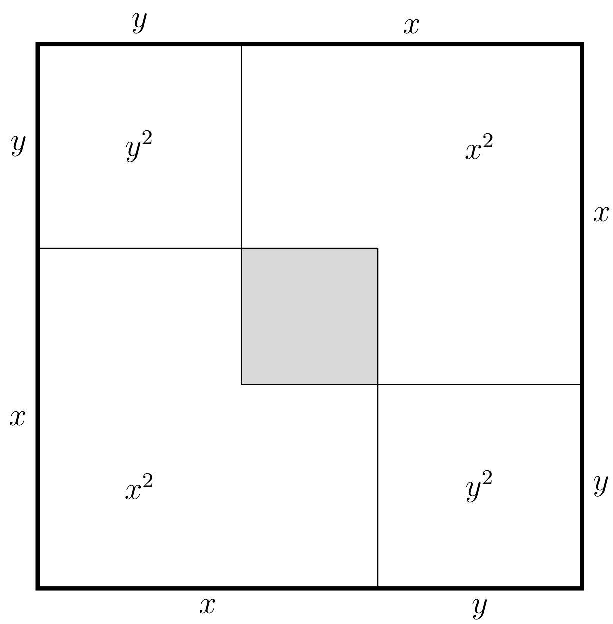

Figure 1. A geometric proof that $(x+y)^2 \le 2(x^2+y^2)$.

Solution. From (3), we have $(b+c)^2 \le 2\left(b^2+c^2\right)$. Multiplying both sides by $a^2\dfrac{b^2+c^2}{2(b+c)^2}$ and then taking the square root yields

$$a\sqrt{\frac{b^2+c^2}{2}} \le a\frac{b^2+c^2}{b+c} = a\frac{(b+c)^2-2bc}{b+c} = a(b+c) - \frac{2abc}{b+c}.$$

Cyclically permuting a, b, and c gives similar inequalities.

Commentary. Once again, we see that a complicated expression is made simpler by observing the cyclic symmetry. This is a good first hint for students. Many of our students still get stuck because they don't see how to employ (3), particularly since the inequality in (3) appears to be in the wrong direction for this application. Give them a chance to be clever. If they need it, remind them that they know how to multiply through an inequality.

Summing these inequalities, we get

$$a\sqrt{\frac{b^2+c^2}{2}}+b\sqrt{\frac{c^2+a^2}{2}}+c\sqrt{\frac{a^2+b^2}{2}} \leq 2(ab+bc+ca)-2abc\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right).$$

Here we use Engel's lemma and the assumption that $a+b+c=3$ to obtain

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{(1+1+1)^2}{2(a+b+c)} = \frac{3^2}{2\cdot 3} = \frac{3}{2}.$$

This estimate implies that

$$a\sqrt{\frac{b^2+c^2}{2}}+b\sqrt{\frac{c^2+a^2}{2}}+c\sqrt{\frac{a^2+b^2}{2}} \leq 2(ab+bc+ca)-2abc\left(\frac{3}{2}\right)$$

as required. □

Observe that, with just these very fundamental tools, we already have students discussing and understanding solutions to genuine Olympiad problems! Let's go further and finish the section with Problem 11670 from the *American Mathematical Monthly* [3]. This problem was proposed by the CSU Fullerton undergraduate students Miranda Bakke and Benson Wu, and by the third author of the present article. It was a result of conversations in a Learning Inequalities Seminar for undergraduate students that ran in parallel to the FMC. The original statement the authors proposed to the *Monthly* was the following.

Problem 6. *([3])*

(a) *Prove that for any $a, b, c > 0$, we have:*

$$\frac{a+b+c}{2} \geq \frac{ab}{a+b}+\frac{ac}{a+c}+\frac{bc}{b+c},$$

with equality if and only if $a=b=c$.

(b) *Prove that, for any $a_1, a_2, \ldots, a_n > 0$, we have*

$$\frac{(n-1)}{4}\left(\sum_{i=1}^{n} a_i\right) \geq \sum_{1\leq i<j\leq n} \frac{a_i a_j}{a_i+a_j},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

The solution published by the AMM applies the Harmonic Means Inequality, which we do eventually explore with the FMC, but here is a more *fundamental* proof.

Solution. As (a) is a special case of (b), we will only give the proof of (b).

(b) Multiplying the inequality in (b) by 2, moving all of terms to the left and then adding $\frac{(n-1)}{2}\left(\sum_{i=1}^{n} a_i\right)$ to both sides gives:

$$(n-1)\left(\sum_{i=1}^{n} a_i\right) - \sum_{1\le i<j\le n} \frac{2a_ia_j}{a_i+a_j} \ge \frac{(n-1)}{2}\left(\sum_{i=1}^{n} a_i\right). \tag{4}$$

Working with just the left-hand side, we regroup the summands to obtain

$$\begin{aligned}\sum_{1\le i<j\le n}\left(a_i - \frac{2a_ia_j}{a_i+a_j} + a_j\right) &= \sum_{1\le i<j\le n} \frac{a_i^2+a_j^2}{a_i+a_j} \\ &= \sum_{1\le i<j\le n}\left(\frac{a_i^2}{a_i+a_j} + \frac{a_j^2}{a_i+a_j}\right) \\ &\ge \frac{[(n-1)\sum a_i]^2}{2(n-1)\sum a_i} \\ &= \frac{n-1}{2}\sum_{i=1}^{n} a_i,\end{aligned}$$

where the inequality above comes from Engel's lemma. This proves (4). The equality case follows from Engel's lemma since it requires $\frac{a_i}{a_i+a_j} = \frac{a_j}{a_i+a_j}$ for all $1 \le i < j \le n$. These equalities imply $a_1 = \cdots = a_n$. □

4 The AM-GM inequality

We have presented the material in this section multiple times in many different ways as part of the Fullerton Mathematical Circle. The approach below reflects the presentation we found most compelling to the students. In this section, we present the gentle proof of the AM-GM inequality we show students, and then several problems that can be tackled through application of the inequality or the method of the proof.

4.1 An interactive lecture

We start our exploration with two important definitions.

Definition 4.1. For positive real numbers $x_1, x_2, \ldots, x_n$, their *arithmetic mean* is defined to be:

$$a = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

and their *geometric mean* is defined to be:

$$g = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

At the beginning of the 19th century, Augustin-Louis Cauchy [7] had established that $g \leq a$. This result is today known as the AM-GM inequality. We present here a proof by induction of the general AM-GM inequality, that will set the stage for a proof of the more general Ladder of Means theorem (but that's a story for another article). Our viewpoint is much indebted to the presentation from P. P. Korovkin [13], which represents, in our view, an expository masterpiece. We start with the following.

Theorem 3. *If $x_1, \ldots, x_n$ are positive real numbers satisfying $x_1 x_2 \cdots x_n = 1$, then*

$$x_1 + x_2 + \cdots + x_n \geq n.$$

Furthermore, equality holds if and only if all of the x_i are equal.

Proof. We proceed by induction on n.

> *Commentary.* Once again, if you find your students not ready to grapple with induction, you can follow the proof below replacing the general case with $n = 2$ and $n = 3$. This should be enough for students to see the pattern.

The result holds trivially when $n = 1$. We now assume the result holds for some $n \geq 1$, and that $x_1, \ldots, x_{n+1}$ are positive real numbers such that $x_1 \cdots x_{n+1} = 1$. If all of these numbers are equal, then they equal 1 and their sum is $n + 1$, proving the result. Otherwise, there exist at least two different numbers and at least one of them is less than 1 and at least one is greater than 1. Without loss of generality, we assume that $x_1 < 1 < x_2$. It follows that the product of the n numbers $(x_1 x_2), x_3, \ldots, x_{n+1}$ is 1, and hence their sum is at least n. Thus,

$$\begin{aligned} x_1 + \cdots + x_{n+1} &= x_1 + \cdots + x_{n+1} + x_1x_2 - x_1x_2 \\ &\geq n + x_1 + x_2 - x_1x_2 \\ &= (n+1) + x_1 + x_2 - x_1x_2 - 1 \\ &= (n+1) + (1-x_1)(x_2-1). \end{aligned}$$

As $x_1 < 1 < x_2$, it follows that $(n+1) + (1-x_1)(x_2-1) > n+1$, completing the proof. □

We now prove the AM-GM inequality.

Theorem 4 (AM-GM inequality)**.** *If $x_1, x_2, \ldots, x_n$ are positive real numbers, then their arithmetic mean,*

$$a = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

is at least as large as their geometric mean

$$g = \sqrt[n]{x_1x_2\cdots x_n},$$

i.e., $a \geq g$. Furthermore, equality holds if and only if all of the x_i are equal.

Proof. Raising the geometric mean to the nth power, we have $g^n = x_1 \cdots x_n$, and therefore

$$1 = \frac{x_1}{g} \cdots \frac{x_n}{g}.$$

By Theorem 3, since we have a product of n positive numbers whose product is 1, it follows that their sum is at least n, i.e.,

$$\frac{x_1}{g} + \frac{x_2}{g} + \cdots + \frac{x_n}{g} \geq n,$$

with equality if and only if all of the x_i are equal. Multiplying by g and dividing by n now yields the AM-GM inequality:

$$a = \frac{x_1 + x_2 + \cdots + x_n}{n} \geq g.$$ □

4.2 Some problems

Next, we present some problems whose solutions can use Theorem 3 or the AM-GM inequality. Many of these are examples of extrema problems for quantities with several variables. As in the previous section, we prefer algebraic arguments to those using calculus.

Problem 7. *If $x_1, x_2, \ldots, x_n$ are positive real numbers, then*

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n,$$

with equality precisely when

$$x_1 = x_2 = \cdots = x_n.$$

Solution. Since

$$\frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdots \frac{x_{n-1}}{x_n} \cdot \frac{x_n}{x_1} = 1,$$

Theorem 3 implies that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n,$$

and equality holds only precisely when

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \cdots = \frac{x_{n-1}}{x_n} = \frac{x_n}{x_1} = 1,$$

i.e., when $x_1 = x_2 = \cdots = x_n$.

> *Commentary.* Any trouble here generally arises from the 'Statements' or 'Routine Applications' categories discussed in Section 2, and are easily overcome by checking in with students or prompting discussion within a group.

□

The following few problems are now within reach for our students, often without the hints included here.

Problem 8. *Prove that, for any real number x, we have:*

$$\frac{x^2 + 2}{\sqrt{x^2 + 1}} \geq 2.$$

Hint. The left-hand side term can be expressed as

$$\sqrt{x^2+1}+\frac{1}{\sqrt{x^2+1}}.$$

Now denote $\sqrt{x^2+1}=A$ and think of a simple application of the AM-GM inequality. Students can often be led to this form by reminding them of the previous problem. The application of the AM-GM inequality to a sum of reciprocals is a fiendishly powerful tool.

Problem 9. *Prove that, if $a>1$, then*

$$\log_e(a)+\log_a(e)\geq 2.$$

Hint. Write $x=\log_e(a)$ and note that $\frac{1}{x}=\log_a(e)$.

Problem 10. *Prove that, for any real number x, we have:*

$$\frac{x^2}{1+x^4}\leq\frac{1}{2}.$$

Hint. Note that the left-hand side term can be rewritten as

$$\frac{1}{\frac{1+x^4}{x^2}}=\frac{1}{\frac{1}{x^2}+x^2}.$$

The solution to the following *Gazeta* problem was found by Edward Zeng, a 12th-grader at the time he provided the solution. Edward presented this solution, along with solutions to other problems, as part of a poster presented at the Southern California-Nevada sectional meeting of the MAA in 2017.

Problem 11. *([4]) Prove that, for any $x,y,z\in(0,\infty)$, we have the inequality*

$$\sqrt[6]{(2x+y)(x+2y)(2y+z)(y+2z)(z+2x)(x+2z)}\leq x+y+z.$$

Solution. (*Edward Zeng*) By using the AM-GM inequality, we have the following:

$$\begin{aligned}x+y+z&=[6x+6y+6z]/6\\&=[(2x+y)+(x+2y)+(2y+z)+(y+2z)+(z+2x)+(x+2z)]/6\\&\geq\sqrt[6]{(2x+y)(x+2y)(2y+z)(y+2z)(z+2x)(x+2z)}.\end{aligned}$$

□

Several of our advanced students noted that the following problem showed up in their Calculus textbooks (along with many similar problems). They were delighted to find out that they could solve these problems without any Calculus.

Problem 12. *From all rectangular boxes with a given sum of the three mutually perpendicular edges, find the box with the greatest volume.*

Solution. Suppose that three mutually orthogonal sides have lengths a, b, and c, and that $a+b+c=m$. The volume of the box is given by $V=abc$. The AM-GM inequality implies that

$$V = abc \leq \left(\frac{a+b+c}{3}\right)^3 = \left(\frac{m}{3}\right)^3 = \frac{m^3}{27},$$

where equality holds precisely when $a=b=c=\frac{m}{3}$. □

It is also useful to note that that the AM-GM inequality can be used to analyze transcendental sequences, and even to solve some transcendental equations with just as much ease.

Lemma 4. *For any positive numbers a, b with $a \neq b$ we have:*

$$ab^n < \left(\frac{a+nb}{n+1}\right)^{n+1}.$$

Proof. The argument relies on a direct application of the AM-GM inequality:

$$\sqrt[n+1]{ab^n} = \sqrt[n+1]{abb\cdots b} < \frac{a+b+b+\cdots+b}{n+1} = \frac{a+nb}{n+1}.$$

Note that the inequality must be strict. □

Problem 13. *Show that Euler's sequence is increasing:*

$$x_n = \left(1+\frac{1}{n}\right)^n.$$

Solution. By Lemma 4, with $a=1$ and $b=1+\frac{1}{n}$, we have

$$x_n = \left(1+\frac{1}{n}\right)^n < \left(\frac{1+(n+1)}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = x_{n+1}. \quad \square$$

Problem 14. *Show that the following sequence, closely related to Euler's sequence, is increasing:*

$$z_n = \left(1 - \frac{1}{n}\right)^n.$$

Hint. In Lemma 4, take $a = 1$ and $b = 1 - \frac{1}{n}$.

Problem 15. *Show that the following sequence, closely related to Euler's sequence, is decreasing:*

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

Solution. We connect the values y_n and, using the notation from Problem 14, z_{n+1} as follows:

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} = \frac{1}{\left(\frac{n}{n+1}\right)^{n+1}} = \frac{1}{\left(1 - \frac{1}{n+1}\right)^{n+1}} = \frac{1}{z_{n+1}}.$$

Since (z_n) in increasing, it follows that (y_n) is decreasing. □

Problem 16. *Euler's sequence is bounded.*

Solution. Using the notation from Problem 13 and Problem 15, since (x_n) is increasing, (y_n) is decreasing, and $x_n < y_n$ for all n, it follows that, for all m, n, $x_m < y_n$. Since $y_1 = 4$, it follows that $x_n < 4$ for all n, and (x_n) is bounded. □

Advanced Note: In the solution above, we chose y_1 for simplicity. Alternatively, we could have chosen $y_5 < 3$ to get an even smaller (integer) upper-bound on Euler's sequence. Most readers will recognize that Problem 13 and Problem 16 can be combined to show that Euler's sequence converges to a real number, *Euler's number* e. A rigorous proof of this last step is generally beyond the scope of our Math Circle discussions, but an intuitive explanation of the fact that a bounded, increasing sequence converges is usually enlightening, particularly for those students for whom e has been introduced in school, but with little justification.

Problem 17. *([1]) Solve the equation*

$$2^x + 2^{-x} = 2\cos\left(\frac{x}{3}\right).$$

Solution. By applying the AM-GM inequality, we have the following sequence of inequalities:

$$2^x + 2^{-x} \geq 2\sqrt{2^x \cdot 2^{-x}} = 2 \geq 2\cos\left(\frac{x}{3}\right).$$

These inequalities hold true for all $x \in \mathbb{R}$, with equality only when $2^x = 2^{-x}$, i.e., when $x = 0$. □

Much more sophisticated problems are also accessible using the techniques discussed.

Problem 18. *Prove that for any natural number $n \geq 2$ the following inequality holds:*

$$\sum_{k=2}^{n} \frac{1}{\sqrt[k]{(2k)!}} > \frac{n-1}{2n+2}.$$

Solution. By the AM-GM inequality, we have

$$\begin{aligned}
\sqrt[k]{(2k)!} &= \sqrt[k]{1 \cdot 3 \cdot 5 \cdots (2k-1)} \cdot \sqrt[k]{2 \cdot 4 \cdot 6 \cdots (2k)} \\
&< \frac{1+3+5+\cdots+(2k-1)}{k} \cdot \frac{2+4+6+\cdots+2k}{k} \\
&= \frac{k^2}{k} \cdot \frac{k(k+1)}{k} = k(k+1).
\end{aligned}$$

Commentary. We expect students to get stuck at the beginning of this problem for three main reasons. First, many will be familiar with but still uncomfortable with summation notation. Including a problem in lecture that utilizes summation notation can be helpful. The second difficult part is that the inequality looks very complicated, with a sum and a product. We remind our students of other instances where they had sums and analyzed the terms piece-by-piece. The third point where students tend to stumble is the factorial, which many students will distribute to get $2!k!$. Part of the inspiration for this misstep is that they are trying to use the AM-GM inequality which, since there is a k-th root, wants to have a product with k factors. We simply ask them to check if it is true when $k = 3$. It isn't hard for students to find a way to break the product up into two pieces with k factors. However, the sequential partition will not reduce as nicely as the odd-even partition we use below. Nonetheless, we let them head down the wrong path first as overcoming frustration is part of the game. If students are not familiar with the sum of consecutive odd/even integers, then more guidance is certainly necessary.

The inequality is strict since the integers in the product are all distinct. We conclude that, for all $k \geq 2$, we have

$$\frac{1}{\sqrt[k]{(2k)!}} > \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Recognizing the "telescopic sum" (a term students connect with even if it is unfamiliar), we finally have

$$\sum_{k=2}^{n} \frac{1}{\sqrt[k]{(2k)!}} > \sum_{k=2}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2(n+1)}. \qquad \square$$

The solution to the next problem is another due to Alvin Kim.

Problem 19. *([18]) Find all nonzero natural numbers a, b, c, d such that*

$$a + b + c + d + bcd + acd + abd + abc = 8\sqrt{abcd}$$

Solution. (*Alvin Kim*) Since all the terms are positive, we can use the AM-GM inequality for the terms a, b, c, d, bcd, acd, abd, abc. Therefore, we have

$$\begin{aligned} \frac{a + b + c + d + bcd + acd + abd + abc}{8} &\geq \sqrt[8]{(abcd)^4} \\ &= \sqrt{abcd}, \end{aligned}$$

or

$$a + b + c + d + bcd + acd + abd + abc \geq 8\sqrt{abcd}.$$

Since equality holds only when all of the terms a, b, c, d, bcd, acd, abd, abc are equal, the only solution is when $a = b = c = d = 1$. $\square$

5 Advanced Problems

For students who come regularly or have a stronger background to begin with, it is also helpful to have advanced problems that require a bit more ingenuity in mixing or iterating the methods developed over several sessions.

A challenging question, for example, that receives a lot of attention when students prepare for the mathematical olympiads is the following. Regular attendees will be familiar with both Engel's lemma and the AM-GM inequality.

Problem 20. *([12]) Let $a, b, c > 0$, such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(b+a)} \geq \frac{3}{2}.$$

Solution. Thinking ahead to try to use Engel's lemma, we rewrite each of the summands on the left to put a square in the numerator:

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(b+a)}=\frac{\left(\frac{1}{a}\right)^2}{a(b+c)}+\frac{\left(\frac{1}{b}\right)^2}{b(c+a)}+\frac{\left(\frac{1}{c}\right)^2}{c(b+a)}.$$

To simplify matters, we make the substitutions $a=\frac{1}{x}$, $b=\frac{1}{y}$, and $c=\frac{1}{z}$. Since $abc=1$, we get $xyz=1$ and the AM-GM inequality yields

$$\frac{x+y+z}{3}\geq\sqrt[3]{xyz}=1. \tag{5}$$

After the substitution, we have

$$\frac{x^2}{\frac{1}{x}\left(\frac{1}{y}+\frac{1}{z}\right)}+\frac{y^2}{\frac{1}{y}\left(\frac{1}{x}+\frac{1}{z}\right)}+\frac{z^2}{\frac{1}{z}\left(\frac{1}{x}+\frac{1}{y}\right)}=\frac{x^2(xyz)}{y+z}+\frac{y^2(xyz)}{x+z}+\frac{z^2(xyz)}{x+y}$$
$$=\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}.$$

By Engel's lemma and (5), we have

$$\begin{aligned}\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}&\geq\frac{(x+y+z)^2}{2(x+y+z)}\\&=\frac{x+y+z}{2}\\&=\frac{x+y+z}{3}\cdot\frac{3}{2}\\&\geq\frac{3}{2},\end{aligned}$$

completing the proof. We also note, from the use of the AM-GM inequality, that equality occurs precisely when $a=b=c=1$. □

Although still a year or more away from taking the exam, our students are even being trained to solve Putnam problems—with short, elegant solutions—using only the elementary methods described above!

Problem 21. *([16]) Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be nonnegative real numbers. Show that*

$$(a_1a_2\cdots a_n)^{1/n}+(b_1b_2\cdots b_n)^{1/n}\leq[(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

Solution. Assume without loss of generality that $a_i + b_i > 0$ for each i (otherwise both sides of the desired inequality are zero). The AM-GM inequality implies that

$$\left[\frac{a_1}{(a_1+b_1)}\cdots\frac{a_n}{(a_n+b_n)}\right]^{1/n} \le \frac{\left(\frac{a_1}{a_1+b_1}+\cdots+\frac{a_n}{a_n+b_n}\right)}{n}.$$

Similarly,

$$\left[\frac{b_1}{(a_1+b_1)}\cdots\frac{b_n}{(a_n+b_n)}\right]^{1/n} \le \frac{\left(\frac{b_1}{a_1+b_1}+\cdots+\frac{b_n}{a_n+b_n}\right)}{n}.$$

Adding these two inequalities gives

$$\left[\frac{a_1}{(a_1+b_1)}\cdots\frac{a_n}{(a_n+b_n)}\right]^{1/n} + \left[\frac{b_1}{(a_1+b_1)}\cdots\frac{b_n}{(a_n+b_n)}\right]^{1/n} \le 1$$

Clearing denominators now yields the desired result. □

6 Conclusion

Taking inspiration from G. Ţiţeica's discovery of the 11th grade student Dan Barbilian, through his stellar problem skills at a fundamental level [6], our Math Circle's fundamental philosophy is that good problems are an avenue and an introduction into mathematical research. As such, we see the above problems as much more than extensions of the traditional algebra curriculum designed to challenge our students and exercise their analytical skills. There are natural extensions of these ideas that lead into research mathematics, as is illustrated in, for example, [20, 21, 22]. In fact, a module on the foundations of inequalities was part of FMC alum Bryan Brzycki's training in 11th grade, when he participated in the USA Mathematical Olympiad. While training for the USAMO, Bryan was presented with a challenge related to a research project in differential geometry. That particular challenge had an elementary flavor quite suitable for his background and his skills. Bryan proved the statement, and his proof made it to print in [5].

The structure of this program is highly flexible and adaptable, but has a clear trajectory that can and has led to amazing projects. We have several examples of students rising up through our program to engage fully in research mathematics. Algebraic inequalities in conjunction with a few classical results provide an easy access point, while the methods of application provide for a full range of skill development perfect for the Math Circle setting.

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