Journal of Math Circles

Volume 2 | Issue 1 Manuscript 1032

January 2021

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Recommended Citation

Long, Jane H. and Richardson, Clint (2021) "A Math Without Words Puzzle," *Journal of Math Circles*: Vol. 2 : Iss. 1 , Article 5.
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A *Math Without Words* Puzzle

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A visual puzzle by James Tanton forms the basis for a session that has been successfully implemented with middle and high school teachers, students in grades 4-8, undergraduate mathematics majors, and served as a demonstration session for Math Circle events at conferences. Designed to be presented with no directions or description, the puzzle requires participants to discover the goals themselves and to generate their own questions for investigation. Solutions, significant facilitation suggestions, and possibilities for deep mathematical extensions are discussed; extensive illustrations are included.

Keywords: All levels, Graph Theory, Parity, Visual Puzzle, Without Words

1 Introduction

We describe a session exploring a puzzle from James Tanton's *Without Words* [5]. As the title indicates, no instructions or descriptions are provided. According to Tanton, this volume is a collection of

> ...immediately accessible but deeply mathematical puzzles, all designed to offer true joy in thinking mathematically in creative, innovative and surprising ways...These puzzles are universal: they transcend the barriers of language and culture, literally, and are thereby accessible to all people on this globe.

Here, the absence of written directions represents a lack of boundaries or barriers, and affords the opportunity to develop ownership of mathematical ideas and explore them to the depth and breadth desired.

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This puzzle is a true "low entry, high ceiling" problem that has field-tested well in sessions for different audiences and time frames: student circles, teaching conferences, teachers' circles, and a senior undergraduate capstone project. Serious mathematics arises at all levels, with mathematical ideas including parity, modular arithmetic, graph theory, combinatorics, and algorithms. Long-term investigations can arise through brief introductions to new subjects and vocabulary. For a meaningful experience, participants need only understand even and odd numbers and be able to sum a list of several numbers. This activity is a good "ice breaker": an early-in-the-year session to encourage participants to begin to communicate and collaborate. This also means that the activity is well-suited for a meaningful stand-alone session or demonstration.

1.1 The Puzzle

Intrepid readers and aspiring Math Circle session facilitators are strongly encouraged to spend time with this puzzle before reading further.

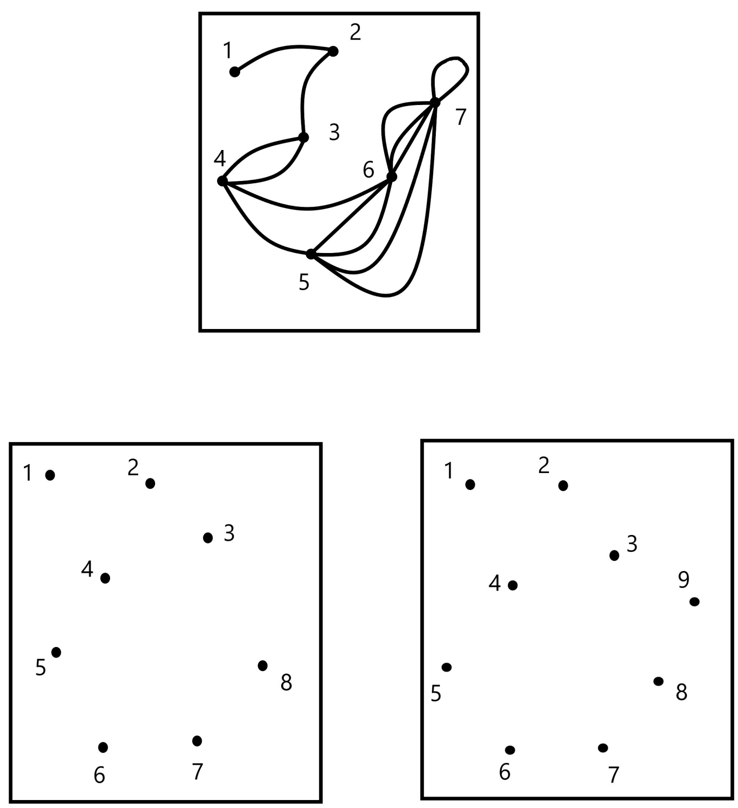

Figure 1. A Math Without Words Puzzle [5]

In this unique problem, the first step to a solution involves explicitly stating a goal or question. Initially, to most participants, the graphical presentation naturally suggests a two-part task: first, determine what is happening in the top drawing and, second, figure out how to do the same thing in the next two drawings. It turns out, however, that the second part of the task amounts to figuring out *if* it is possible to do the same thing in the next two drawings.

1.2 An Initial Solution

Readers who have considered and played with the puzzle are invited to read further.

The goal is to discern the "rule" that the first drawing follows and to complete the other drawings according to that same rule by connecting the dots correctly, if possible. The rule is generally interpreted as: *the number on each vertex counts the number of edge endpoints connected to that vertex.*[1] Notice that the loop at vertex 7 contributes 2 to the total:

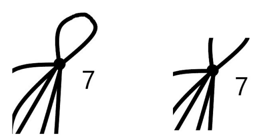

Figure 2. Zooming in to Explain Vertex Labels

When the number of edge endpoints connected to a particular vertex matches its label, we will say that the vertex is satisfied. Throughout, n denotes the number of vertices in a drawing.

Readers familiar with graph theory will recognize the vertex label as its *valence, valency,* or *degree*; this and other graph theory ideas will be further explored in subsection 4.3.

Figure 3 shows a completed drawing[2] with 8 vertices—that is to say, a drawing where each vertex is satisfied. While other solutions are possible for the case of 8 vertices, it is important to note that a complete drawing in the case of 9 vertices is impossible.

[1] Facilitators should use their judgment in choosing the wording *dot* (which is most appropriate for students in grades 4-12), *vertex*, or *node*. Following conventions of graph theory, we use the word *vertex* and call the connecting curves between vertices *edges*.

[2] The terminology *complete(d) drawing* was chosen to avoid mis-characterizing graphs as *complete graphs*, a term with a distinct mathematical meaning.

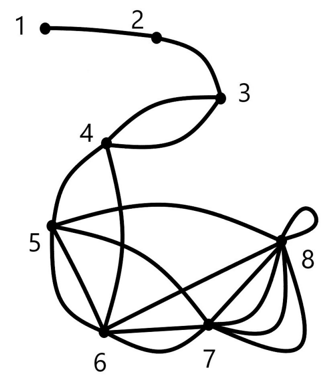

Figure 3. A completed drawing with 8 vertices

Since each edge contains exactly two endpoints, the number of endpoints in a completed drawing must be even. This condition corresponds to the sum of the vertex labels being even; that is, the summation

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

needs to be even. As it turns out, the sum is even if and only if n is equivalent to 0 or 3 modulo 4 (see subsection 4.1), and it is possible to build solutions for every such value of n (see Appendix A).

If the group agrees to allow disconnected drawings, they may also present the visual argument in Figure 4. Note that this diagram suggests a strategy to extend a solution for 7 vertices to a solution for 8 without much effort. This approach only applies at an even vertex as the 'new' vertex is satisfied using loops–edges with both endpoints at the same vertex.

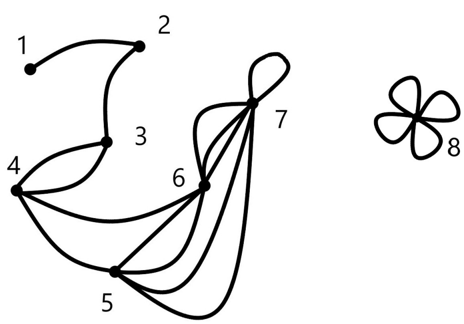

Figure 4. A visual argument for a disconnected solution

2 Implementation Considerations

Very briefly, the suggested format for a lesson is to give each participant a copy of the puzzle, explain that there are no instructions so they must determine the goals themselves, and give some quiet think-time before bringing the group together for discussion. From that point, the facilitator should use their best judgment to incorporate shifts between large group discussion and small group/individual work time; visiting groups and/or individuals to determine progress is strongly recommended, as is finding volunteers to go to the board to explain their thinking. Early in the session, prioritize board work showing solutions for 8 vertices and the 'Yes/No' table (figure 6 in subsection 2.3). To their level of comfort, the facilitator should allow participant-generated questions to direct the rest of the session. Prepare for the session by considering multiple good stopping points; section 4 presents several pathways for deep mathematical exploration and subsection 2.2 gives suggestions for facilitating this session with students in grades 4-8.

Facilitators should watch for participants who do not seem to be able to make a start on the puzzle. Once the meaning of vertex labels has been established, these participants may have an entrée to the problem through verifying the rule for the completed drawing very carefully. They may also appreciate being asked to verify solutions for 8 vertices as others add them to the board.

In a group setting, the fact that there are no instructions means that the group decides what is "allowed" or "not allowed." According to the preference of the facilitators, these decisions represent opportunities to keep the group moving in one direction or split into multiple directions. For example, if the question of whether or not loops at a vertex are allowed, the presence of a loop in the original completed drawing with 7 vertices should probably prompt the group to allow them. But if participants wonder whether to permit multiple loops per vertex, it makes sense to allow subgroups of participants to consider either possibility.[3]

A note on teaching assistants: For best results, TAs should try the problem on their own before facilitating or viewing a solution. If a guest facilitator is presenting, sending the problem and a brief solution in separate documents in advance of the session is recommended. In cases where advance preparation is not feasible, it is recommended that the facilitator start the session and then pull TAs for a quick side conversation: at a minimum, discuss with them the meaning of vertex labels, the fact that 7 and 8 are possible but 9 is impossible,

[3]In practice, this question comes up in every session, with any audience.

and that they should prompt students to carefully count endpoints to verify any completed drawings they propose.

2.1 Initial Investigation

Here, we explore a natural progression of mathematical ideas encountered in the solution of this puzzle. Some considerations for different audiences appear here, with additional considerations for younger audiences in subsection 2.2.

The first and most important idea is the meaning of vertex labels. In most sessions, a participant realizes this within 5 minutes or so, and it is recommended that the facilitator ask them to explain the discovery to the group. The facilitator can then take on a secondary role by confirming that the participant's discovery is correct. If participants do not discover the meaning of vertex labels on their own, it would be expedient for the facilitator to ask a question such as, "Do you think the numbers labeling the vertices mean anything?" or to directly reveal the meaning before waiting too long. Focusing on vertex 7 is a helpful starting point because it illustrates how a loop contributes 2 to the total; consider asking participants to zoom in on the vertex 7 in the given completed drawing to explain the vertex labels, as in Figure 2 in subsection 1.2. In the authors' experience, the fact that a loop contributes 2 to the vertex label endpoint count is a subtle thing for some solvers. A large-group discussion clarifying this point helps participants cement both their understanding and the language they can use to communicate the "rule" for satisfying vertices in completing a drawing.

Once the meaning of the vertex labels is clear, participants can begin work to complete the puzzles with 8 and 9 vertices. If frustration occurs, the facilitator might ask participants to try completing a drawing with fewer vertices. (If participants have trouble with 1 or 2 vertices, which are not possible, suggest they try 3 or 4, which are.) This task prompts consideration of whether it is possible to complete drawings for any number of vertices and leads nicely to the 'Yes/No' table outlined in Figure 6, section 2.3.

In some sessions, participants want to allow a "hanging edge," where one endpoint of an edge connects to a vertex but the other does not (see Figure 5). This is more likely to occur in young students but has also been observed in elementary or middle-school teachers. It is best to direct young participants to consider the more interesting question of drawings without hanging edges. Otherwise, they could correctly conclude that completed drawings are possible for all n, and the investigation is far less rich. Usually, a group of adults realizes that the hanging-edge situation is less interesting without intervention from

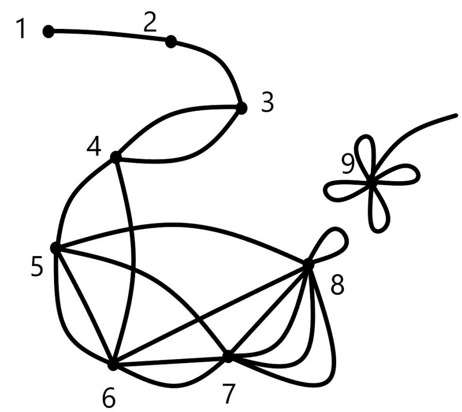

Figure 5. A drawing with a hanging edge at vertex 9, to be avoided

the facilitator, and the group naturally moves to focus on the situation in which hanging edges are prohibited.

On the other hand, some younger participants will immediately start drawing with 8 and 9 vertices without taking time to consider the goal. While it is conceivable that some participants will determine the goal quite quickly, it is best for the facilitator to check in with students who appear to start completing drawings right away.

Also keep in mind that when there is a mistake in drawing edges on a graph, moving a single edge seldom resolves the issue. If participants become frustrated, facilitators might suggest they 1) start over on a fresh drawing, 2) start with a smaller number of vertices, or 3) consider whether they think it is possible to complete a drawing with this number of vertices.

2.2 Considerations for Younger Participants

This session has been successfully implemented with upper-elementary (fourth grade and above) students. The authors suggest the following considerations when working with students younger than high school.

- A bigger graphic can help. Erasures and crowded drawings frequently lead to miscounting. (See appendix C for puzzle handouts.)
- These students often think they have a complete and correct drawing when they do not. As expected, this happens with 9 vertices, but sometimes happens with 8 vertices as well. Facilitators can ask students to show them the endpoint count at each node so they can find the discrepancy themselves.

Journal of Math Circles, Vol. 2, Iss. 1 [2021]

- Younger, less mathematically experienced participants often wholeheartedly believe that a solution does not exist because they tried very hard, but failed to find one. Facilitators may also notice participants pursuing a solution for the drawing with 9 vertices, even after agreeing that completed drawings are impossible if the corresponding sum is odd. This apparent failure to transfer knowledge has been observed in younger students; a facilitator might ask such a student if they always have one "leftover" endpoint.

2.3 Parity and Arithmetic Series

Parity is an essential idea in the solution of this puzzle, but in an unexpected way. As noted in the introduction, completed drawings are possible for 7 and 8 vertices but not for 9. So the parity of n does not determine whether a drawing is possible, but rather the parity of the sum $1 + 2 + \cdots + (n-1) + n$.

This represents a nice opportunity to introduce summation notation (if audience-appropriate) and the arithmetic series formula to students who have not seen it before. Building from a fairly small, concrete example with an even n is a good way to illustrate the $\frac{n}{2}$ pairings that produce $1 + n$:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$$

9

Asking for the sum of the natural numbers from 1 to 100 motivates presenting the more general front-to-back pairing scheme

$$1 + 2 + \cdots + n-1 + n$$

$1 + n$

This leads to the general summation formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

and gives a chance to tell the famous story of Gauss's trick [2]. Even when working with young students, try not to assume that no one in the room has seen the arithmetic series formula. Sometimes students are familiar with it or have heard Gauss's childhood story. Asking these students to explain the formula makes for richer discussion.

Once introduced to the formula, some younger participants (or elementary and middle teachers unfamiliar with the formula) doubt that it is valid for odd n. They express concern about what happens when one of the numbers is without a pair, or that division by 2 will result in a non-whole number. A large-group discussion usually results in someone recognizing that if n is odd, then $n+1$ is even, so that the result will always be an integer. Similarly, they may notice that if the sum is even and the next consecutive integer $n+1$ is even, then the sum $1+\cdots+n+(n+1)$ will also be even. Participants often recognize this on their own, or after being prompted to build and explain a table as in Figure 6. Those experienced with the summation formula almost always realize this without the table, but younger students may need to spend significant time with the table. In fact, creating and discussing a 'Yes/No' table makes a meaningful conclusion for a session for younger students.

Vertices	1	2	3	4	5	6	7	8	9	$\cdots$
Completed Drawing Possible?	No	No	Yes	Yes	No	No	Yes	Yes	No	$\cdots$

Figure 6. A Yes/No Table for Possible Drawings

If asked whether a completed drawing is possible for 100 or 101 vertices, students will sometimes use a modified skip-counting method with the pattern "no-no-yes-yes" in order to make a conclusion.

3 Discussion

Here, we discuss approaches taken by different audiences and highlight this puzzle as an effective way to illuminate mathematical practices of mind. The extensive section 4 presents avenues for deeper and more advanced explorations, including those that may be suited for long-term investigation.

3.1 Observations Concerning Different Audiences

Certainly, investigations of this puzzle will vary based on the participants' level of experience with mathematical language and formal reasoning. Some participants are completely paralyzed by the absence of instructions for the puzzle and lack of direction from a facilitator. In one session of perhaps 15 students in grades 4 through 8, two or three of the students resisted making any drawings or conjectures because they could not see the path to a solution before beginning to write. This paralysis could be a result of experiences that have led participants (of any age or situation) to view mathematics as a procedural and un-creative endeavor, or who are uncomfortable with the idea of productive failure. Since perseverance is one of the learning goals of Math Circles, facilitators should not necessarily seek to protect participants from these experiences but may want to give some thought to providing encouragement or support in the form of gentle hints. Spending additional time discussing the task with students who have a difficult time starting has proven to be helpful, as is modifying the task to consider fewer vertices (3 and 4 first, followed by 1 and 2) or asking students to verify solutions created by others.

In addition to the observations above, the authors noted that different audiences exhibited varying levels of comfort with implied assumptions. Secondary teachers and undergraduate mathematics majors were likely to dismiss the idea of a hanging endpoint almost immediately, while elementary or middle school teachers and students were more likely to want to investigate this scenario. Secondary teachers and undergraduates were more likely to apply tools such as the arithmetic summation and modular arithmetic, and to apply these tools quickly. However, middle school teachers frequently articulated deeper observations related to parity and meaningful statements about conclusions that could, and could not, be drawn from the group's work.

3.2 Mathematical Practices of Mind

While this problem is engaging and enjoyable in and of itself, one of the primary purposes of sessions such as this is to highlight the metacognitive skills and practices of mind essential to solving problems. Sometimes participants are able to internalize these practices without prompting, but the authors recommend explicitly naming and discussing them. In particular, Polya's problem solving method [4]—

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

—is beautifully illustrated in the exploration of this problem. Step 1—Understand the problem—requires more care in this puzzle than in many other mathematical tasks, so this session is a natural place to emphasize its importance.

Other common problem-solving frames arising from this exploration include:

- Solve a simpler problem: fewer vertices
- Look for patterns: arithmetic series formula, 'Yes/No' table (Figure 6)
- What conclusions does our work allow us to make? What conclusions does it not allow us to make?
- What mathematical ideas does this puzzle suggest to you?
- What other questions could we investigate?

As do many other Math Circles, the East Texas Math Teachers' Circle ends each session with a participant-led debrief of problem-solving skills. Regular attendees have the powerful experience of seeing these approaches surface in a variety of problems and mathematical subject areas. Undoubtedly, participants will generate meaningful additions to the list above, especially if such discussion is part of your Math Circle's tradition.

4 Advanced Mathematical Connections and Extensions

A session that includes the discoveries presented in section 2 above should be considered perfectly successful. This section describes deeper mathematical connections which can be explored in multiple sessions or long-term projects. Experienced presenters may want to stop reading here, so that they can let some of the following ideas come up naturally as part of a session and experience them without preconceptions. In the authors' experience, sessions for students in grades 4-8 or elementary teachers are unlikely to reach these extensions in 120-minute sessions. Middle and high school teacher sessions have reached the point of investigating modular arithmetic (4.1), constructive algorithms (A), and equivalent solutions (4.3). The constructive algorithm presented in Appendix A was independently discovered by an undergraduate math major in a semester-long project and by the second author.

4.1 Modular Arithmetic

One reason this puzzle works well for different audiences is that even early elementary students have an everyday, intuitive understanding of even and odd numbers. Introducing the forms of even numbers as $2t$ and odd numbers as $2t+1$ can support precise mathematical communication and lead to the idea of modular arithmetic. If facilitators wish to emphasize modular arithmetic, they should articulate the connection to the remainders of even and odd numbers when divided by 2.

Students have noted that values of n that are "divisible by 4" or "one less than a multiple of 4" yield a sum that is even; facilitators can urge a connection to the remainder when divided by 4. Furthermore, the fact that the period of the 'Yes/No' sequence is 4 leads to investigation of equivalence classes modulo 4. Those comfortable with algebraic manipulation (middle and secondary teachers, secondary students, undergraduate math majors) can use the summation formula with values of n equal to $4m$, $4m+1$, $4m+2$, and $4m+3$, performing algebraic manipulations in order to verify that the sum $\sum_{k=1}^{n} k$ is even if and only if n is equivalent to 0 or 3 modulo 4.

4.2 Logic and Proof

Perhaps the most accessible conclusion to draw from this problem is that in order for a completed drawing to be possible, the sum of the vertex labels must be even. This is a *necessary* condition, but more work must be done in order to determine whether it is a *sufficient* condition. Facilitators can prompt deep discussion by listening to whether participants use language such as, "we can't do it if the sum is odd," which is true, and "we can always do it if the sum is even," which would almost certainly be a conjecture early in a session. Statements such as these are frequently made by students in grades 4-8 but also by adults. Depending on the level of the audience, facilitators can choose to devote time to discussing:

- The process of making and testing conjectures
- Articulating assumptions (example: did the group implicitly assume hanging edges were not allowed, or clearly declare it?)
- The effect of adding or removing restrictions (this is particularly relevant if the group or some subgroup considered hanging edges earlier in the session)
- The difference between finding a result for a specific value of n versus a result that holds for all values of n (and what type of number n is)
- The difference between being unable to find a solution and logically proving that a solution does not exist

4.3 Graph Theory

Graph theory is an accessible subject which is rarely presented in the standard K–16 curriculum, making it an excellent topic for a Math Circle. Teachers (and undergraduate participants) may be familiar with graphing on a coordinate plane but will probably not have thought about graphs in the more abstract sense of connections between objects. Graphs in which distance and location are unimportant will almost certainly be new to them.

One important, big-picture lesson of this problem is that having a consistent, shared vocabulary is essential to communication; making connections from the words created by your groups to the terms that are common in graph theory (*vertex/node*, *edge*, *valence/order of a vertex*, etc.) can help students realize that terminology in mathematics grows organically when one solves problems.

Terminology from graph theory can also be used to describe and classify different types of solutions. As in subsection 1.2, the solutions presented in Figures 3 and 7 are connected graphs while Figure 4 shows a disconnected graph.

Some graph-theoretical investigations can be done quickly, while others would best extend into long-term projects. Here are some good questions to consider, listed in roughly increasing order of difficulty:

(a) Must vertex 1 always connect to vertex 2? *(No)*

(b) Does the placement the vertices matter? *(No)* Could vertices be drawn in a straight line rather than the given 'S'-shape? *(Yes)*

(c) For which n, if any, do disconnected solutions exist? *(All n equivalent to 0 or 3 modulo 4; see Appendix A)*

(d) For which n, if any, do solutions without loops exist?

(e) For which n, if any, do solutions without cycles ([3]) exist?

(f) What makes solutions distinct?

(g) Are different colorings possible for a given completed drawing?[4]

[4] A *graph coloring* is an assignment of colors to the vertices of a graph such that no two vertices sharing an edge (called *adjacent vertices*) are assigned the same color. There are many interesting questions associated to this topic, see [3].

(h) For which n, if any, do planar solutions exist?[5]

Different methods to describe distinct solutions can lead to long discussions. It may be advisable to ask what makes *this* solution different from *that* one, particularly with respect to specific participant-generated examples. Precisely communicating the differences is a meaningful and challenging task: the question of what makes solutions distinct can be quite subtle. For example, the two drawings in Figure 7 should be considered equivalent because they differ only in the placement of one edge connecting vertices 5 to 8. Thus, distinguishing solutions is not necessarily as simple as saying that one has crossings and another does not.[6] Regarding questions about loops and cycles, note that a completed drawing must contain either a loop or a cycle.[7]

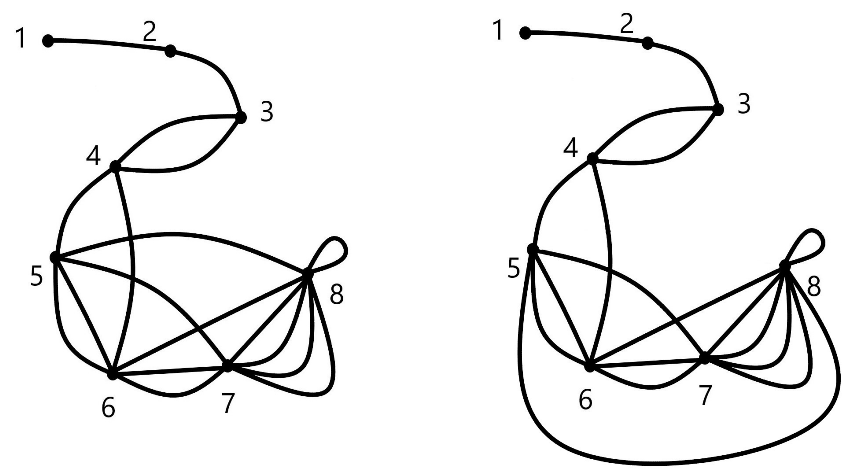

Figure 7. Equivalent Solutions with 8 Vertices

4.4 Combinatorics

Naturally curious Math Circle participants will want to count the things they discover. Results for general n can be difficult to determine, but participants at all levels can create tables counting results for specific values of n and make conjectures in many situations.

[5]A *planar* graph has a projection onto a 2-dimensional plane for which no two edges cross.

[6]In fact, these drawings represent two different projections of the same planar graph. It is reasonable not to bring up the issue, but if it comes up, it would be best to acknowledge that mathematicians who study graph theory would not consider these solutions to be distinct.

[7]In any *simple graph* with at least two vertices, there must be two vertices that have the same degree, so completed drawings in this task are not simple. See [3].

Below are some possible extensions. The first two questions are relatively straightforward, even the generalization of part (a). The remaining questions are more open-ended and turn out to involve quite deep mathematics, so they are more suitable for extended exploration, such as undergraduate research projects. As of yet, the authors have not engaged in such projects beyond the constructive algorithms presented in Appendix A.

(a) How many edges does a completed drawing contain? $(\frac{1+2+\cdots+n}{2})$

(b) In a given completed drawing, how many different paths exist between two given vertices?

(c) Can graph characteristics associated to counting help determine when solutions are distinct?

(d) How many solutions are possible for each n?

(e) How many (connected, disconnected, planar, no-loop,... see subsection 4.3) solutions are possible for a given n?

(f) What is the minimal number of repeated edges in a completed drawing having n vertices?

(g) Can participants use known invariants to distinguish solutions? Can they generate their own invariants?

When counting the number of distinct solutions possible for a given n, facilitators may wish to prompt the observation that distinct drawings can be created by interchanging two repeated edges with two loops, as illustrated in figure 8.

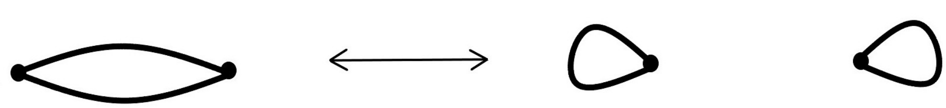

Figure 8. Interchanging Two Repeated Edges with Two Loops

Again, depending on interest and expertise, advanced participants (math majors in particular) could write code to generate and count examples.

Acknowledgments

The authors are extremely grateful to James Tanton for sharing the puzzle and for his encouragement to write this article. The first author thanks Ed Burger for his explanation of the distinction between habits of mind and practices of mind. The editors and reviewers of the Journal contributed many helpful suggestions. The images in this article are modeled after those of Natalya St. Clair.

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A Appendix A: Constructive Algorithms

As mentioned in subsection 4.2, ruling out certain numbers of vertices is not the same as guaranteeing that completed drawings exist for the values which were not ruled out. Mathematically experienced participants may be able to make a constructive algorithm for building solutions. Depending on the interest of participants, the expertise of facilitators, and the time frame, such algorithms could be analyzed. One possible algorithm, presented here, was independently discovered by the second author and an undergraduate math major. The construction uses a recursive procedure to generate future solutions in a 'stem-and-leaves' approach.

Base case:

- ($n = 1$ and $n = 2$) There are no solutions for the cases $n \equiv 1 \mod 4$ and $n \equiv 2 \mod 4$, as discussed in section 4.1; the first two steps of the recursive process below also will not result in solutions, but are necessary to generate the solutions that follow.

- ($n = 3$) A solution can be generated by connecting vertex 1 to 2 with one edge, then 2 to 3 with one edge, then finishing with a loop at 3; this loop provides the remaining two edge endpoints (and is a 'leaf' in the above analogy). See figure 9.

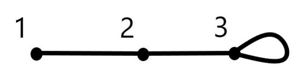


Figure 9. Base case: ($n = 3$) Completed drawing with 3 vertices

- ($n = 4$) A solution can be generated by taking the previous diagram, 'breaking' the loop at 3 and connecting both edges to 4 (which provides two edge endpoints at 4), then finishing the vertex at 4 with a loop to provide the remaining two. See figure 10.

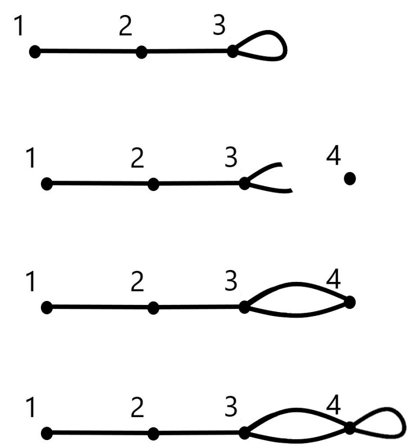


Figure 10. Base case: Generating a Completed Drawing with 4 Vertices

Recursive part: (the following steps can be visualized by referring to vertices 5, 6, 7, and 8, respectively, in Figure 11.)

- ($n \equiv 1 \mod 4$) Note that any solution generated by this algorithm produces a solution diagram having at least one loop at a vertex labeled with a multiple of 4 (0 mod 4). 'Break' this loop and connect the two edges to this vertex, whose label is equivalent to 1 mod 4, and thus is odd. This provides two edges endpoints at this vertex; add loops until a single edge endpoint remains. Connect that edge to the next vertex, which has a label that is equivalent to 2 mod 4. (The diagram that results from this step, of course, does not satisfy the rules and so does not generate a solution, but is necessary to construct the later diagrams which do yield solutions.)

- ($n \equiv 2 \mod 4$) Now, there is one edge connecting to this vertex from the previous step; since the label of the current vertex is equivalent to 2 mod 4, an even number of edge endpoints are needed to satisfy the required number. Add loops until a single edge endpoint is left unsatisfied and connect that edge to the next vertex, which will have a label that is equivalent to 3 mod 4. (This step also does not result in a solution.)

- ($n \equiv 3 \mod 4$) At this stage, we expect a solution. A single edge connects to the current vertex from the previous step. The current vertex has a label that is equivalent to 3 mod 4 and thus is odd; this means that we have an even number of edge endpoints left. Since each loop

Journal of Math Circles, Vol. 2, Iss. 1 [2021]

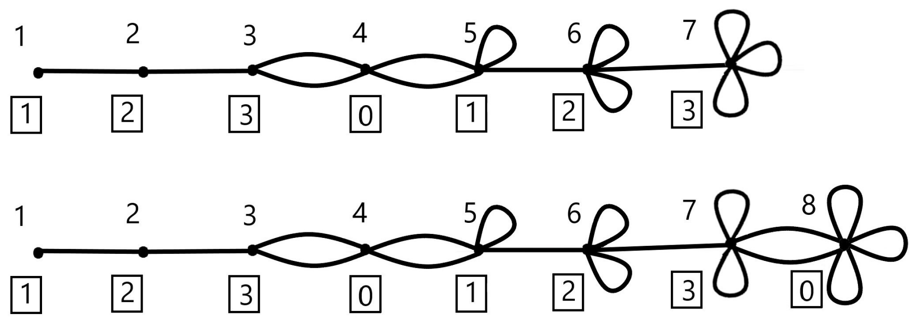


Figure 11. Recursive Step in Constructive Algorithm; equivalency of vertex labels modulo 4 appear within frames

adds an even number of edge endpoints, we can complete the diagram by adding the appropriate number of loops at this vertex, thereby generating a solution. (The first diagram in Figure 11 shows the result for $n = 7$.)

- ($n \equiv 0 \mod 4$) Assuming the conclusion in the previous step, we 'break' one of the loops and connect the resulting endpoints to the current vertex; since the current vertex has a label that is equivalent to 0 mod 4, this label must be even. We already have two edge endpoints, leaving an even number unsatisfied; again, add the appropriate number of loops to add the required number of edge endpoints and thereby generate a solution. (The second diagram in Figure 11 shows the result for $n = 8$.) Finally, note that there is at least one loop at this vertex as was required at the beginning of the recursive part.

If the vertices are arranged in a line, this algorithm produces a connected solution with a main 'stem' connecting the vertices to one another (in the pattern single-single-double-double) along with several 'leaves' (loops) at each vertex, terminating at a vertex with an incoming double-stem and the rest leaves.

Other algorithms are certainly possible; if disconnected solutions are allowed, the algorithms can be even simpler. One can easily generate solutions by completing all even vertices with loops, connecting each vertex of the form $n \equiv 1 \mod 4$ to the associated $n+2$ vertex with a single edge, then finishing these odd vertices with the appropriate number of loops. See figure 12.

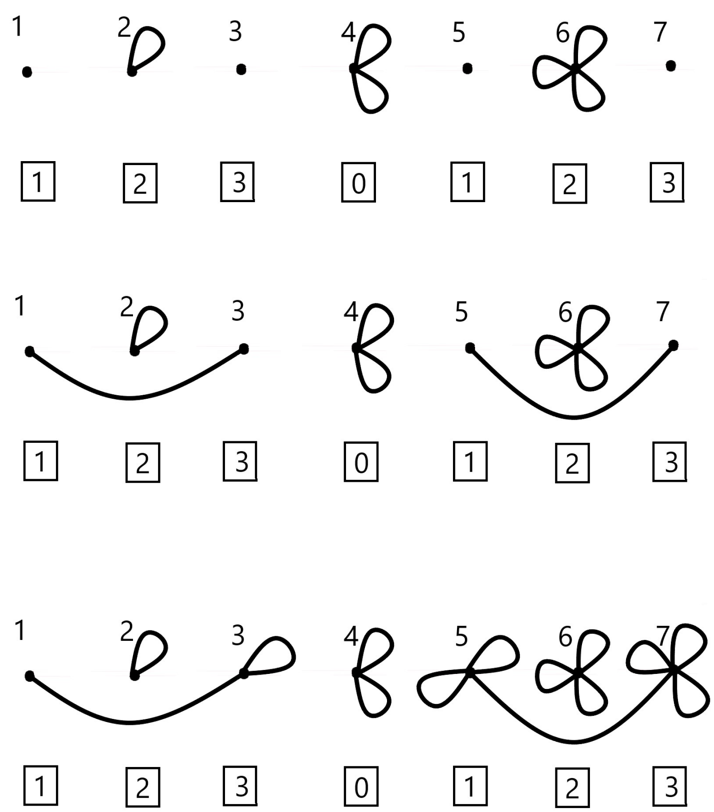


Figure 12. One Algorithm for Constructing Disconnected Solutions

Other observations:

- Given any solution for $n \equiv 3 \mod 4$, add a vertex with $(n+1)/2$ loops to create a solution for $n+1$ vertices; see figure 4 in subsection 1.2.
- One can take any solution for $n = 3$, make a copy of its form and update the numbers on the vertices (say, to 5, 6, 7), then add a pair of loops at each vertex (or four loops at each vertex for 9, 10, 11; six loops for 13, 14, 15; etc.).
- Finally, the vertices with labels equivalent to 0 mod 4 could be finished with loops (or the last step of the 'stem-and-leaf' procedure could be used). See figure 13.

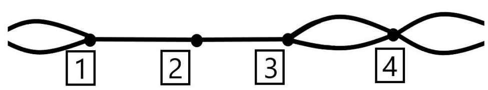


Figure 13. Copying and Modifying a Solution for $n = 3$

B Appendix B: Materials List

This session can be implemented with minimal materials. The authors suggest:

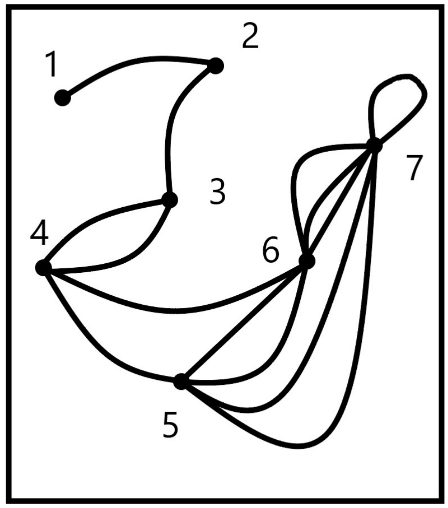
- Many copies of the puzzle (see Appendix C: Puzzle Handout), as many as 3 for each adult participant and 5 or 6 for each child
- Blank paper (as many as 10 pieces per participant)
- Pencils and erasers
- Board space and different-colored writing implements (optional)

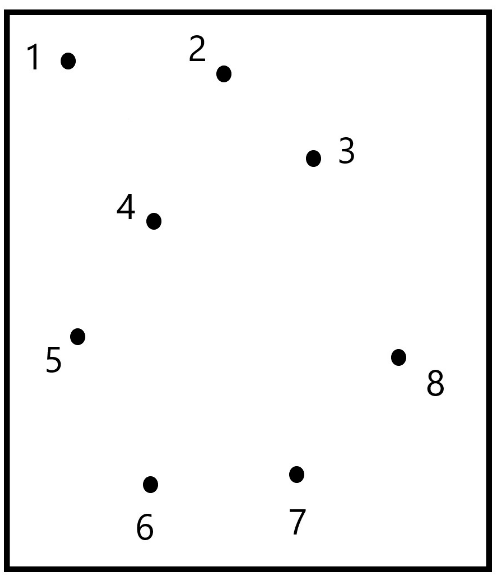
Alternatively, the session could be implemented with transparent plastic sheet covers and dry erase markers. One drawback to this approach is the loss of records of successful and failed approaches.

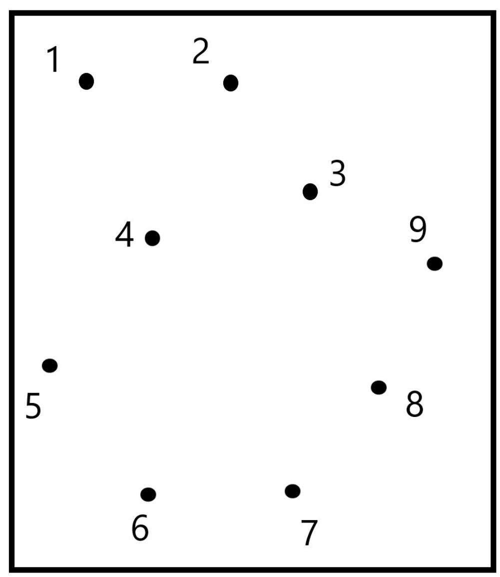
For a session implemented virtually, consider sending the handouts in Appendix C in pdf format to the participants shortly before the session. In advance of the session, suggest participants gather the other materials listed above.

Journal of Math Circles, Vol. 2, Iss. 1 [2021]

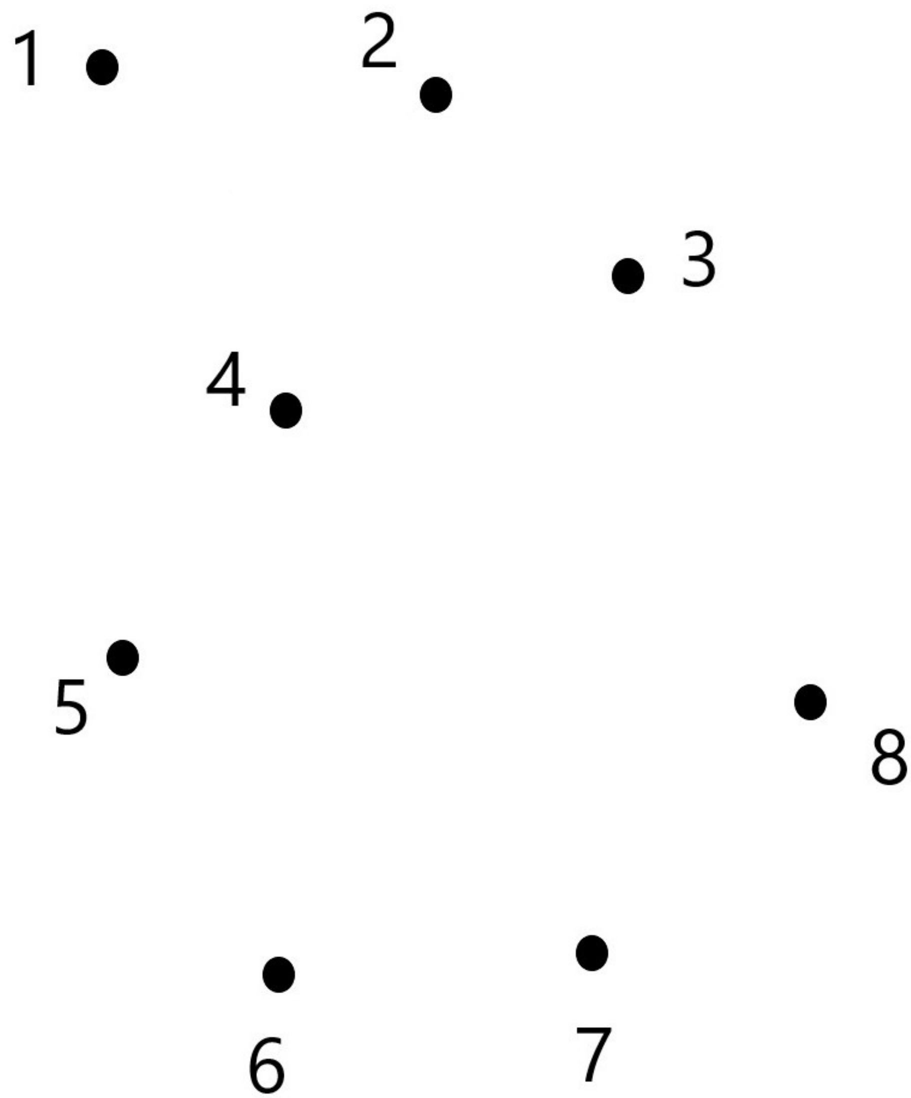
C Appendix C: Puzzle Handouts

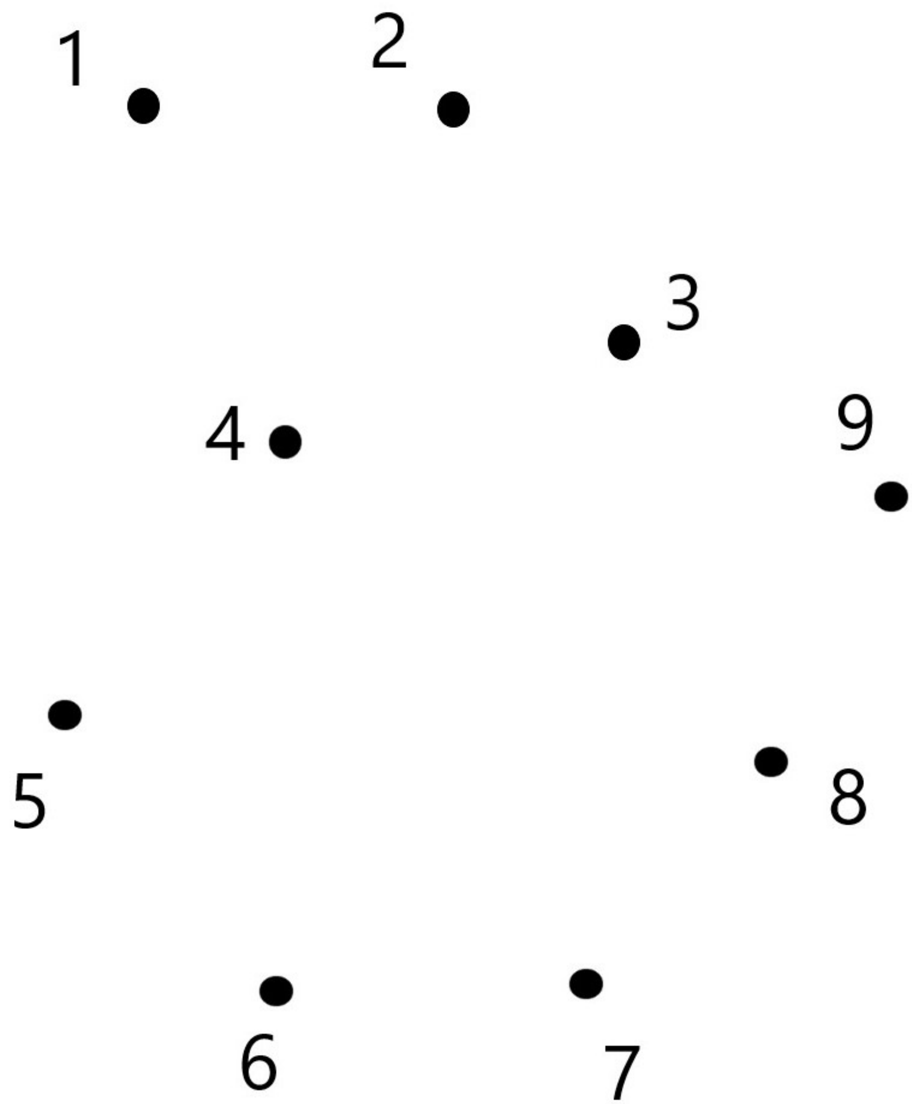






Journal of Math Circles, Vol. 2, Iss. 1 [2021]

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