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A ZETA FUNCTION FOR JUGGLING SEQUENCES

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Abstract

We give a new generalization of the Riemann zeta function to the set of b-ball juggling sequences. We develop several properties of this zeta function, noting among other things that it is rational in b^{-s}. We provide a meromorphic continuation of the juggling zeta function to the entire complex plane (except for a countable set of singularities) and completely enumerate its zeroes. For most values of b, we are able to show that the zeroes of the b-ball zeta function are located within a critical strip, which is closely analogous to that of the Riemann zeta function.

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1. Introduction

This work finds its motivation in two disparate sources. The first is the recognition that in recent years a large and growing body of mathematics has grown out of the study of juggling. Recent work by Chung and Graham [CG] has been devoted to *primitive juggling sequences*, which as (loosely) the building blocks of juggling sequences, serve as a rough analogue to prime numbers among the integers.

The second motivation comes from the awareness that in several branches of number theory and combinatorics, zeta functions are used to study primes and their analogues. Based on the work of Riemann (and ultimately Euler), these functions encode information about all primes (or their analogues), and allow the primes to be studied via analytic techniques. In this paper, we develop a zeta function for primitive juggling sequences, and develop several results about this function.

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2. Background and Notation: Juggling

The notation used for juggling sequences is almost, though not quite, standardized. In the present paper, we will follow the notation used in [CG], which served as a motivation for this work. Jugglers classify juggling patterns using *siteswap* notation, which is based on the height to which each consecutive ball is thrown. "Height" measures the number of beats, or units of juggling time, that a ball is in the air. For example, the sequence $(5, 1)$ indicates that the first ball should be thrown to height 5, and the second to height 1, at which point the pattern repeats. Mathematicians refer to examples of siteswap notation as juggling sequences. Mathematically, a *juggling sequence* is a sequence $T = (t_1, t_2, \dots, t_n)$ for which the values $i+t_i \pmod{n}$ are all distinct. For jugglers, this condition is equivalent to the requirement that no two balls land in the same hand at the same time. We write $t_i = 0$ to indicate that no ball is thrown at time i.

This notation has several interesting properties, one of which is that the average of the entries in the juggling sequence is equal to the number of balls being juggled. Readers are encouraged to consult the comprehensive book by Polster [P] for more interpretations and use of juggling patterns and siteswap notation.

It is interesting to consider what it would mean to do arithmetic on juggling sequences. The most obvious way to combine two sequences is to concatenate them. We shall refer to this as *multiplying* the two sequences. However, this type of multiplication is a bit strange, in that not every pair of sequences can be multiplied. For example, take the first two 3-ball sequences learned by most jugglers: (3) and $(5, 1)$. If we multiply these, we get $(3, 5, 1)$, which is not a juggling sequence (because $3 + 1 \equiv 5 + 2 \pmod{3}$). In order to determine whether two sequences *can* be multiplied, it is necessary only to know the *state* of the sequence—a binary sequence that indicates when the balls currently in the air are going to land. Two juggling sequences can be multiplied if and only if they are in the same state (see [CG, p. 186] for details concerning juggling states). In this paper, we will be using only sequences which have a *ground state*—b-ball sequences with a state consisting of b 1's.

As a converse to multiplication, we note that some juggling sequences can be decomposed (factored) into shorter sequences. For example, the sequence $p = (4, 2, 4, 4, 1, 3)$ can be decomposed into $(4, 2)$, $(4, 4, 1)$, and (3). A juggling sequence is called *primitive* if it cannot be decomposed into shorter sequences. In this way, primitive sequences serve as the analogue to prime numbers in the space of all juggling sequences.

Let us further define $J = J(b)$ to be the set of all ground state b-ball juggling sequences, with $J_P = J_P(b)$ denoting the subset of primitive ground state b-ball juggling sequences.

3. Background and Notation: Zeta Functions

The idea of capturing information about all primes in one function dates most famously to Riemann [R], though it was defined earlier by Euler [E]. As usually written, the zeta function is defined as

$$\zeta(s) \ = \ \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The zeta function has an Euler product, and (in its half-plane of convergence) it can also be expressed as a product over only primes, rather than as a sum over all positive integers:

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \qquad (\text{for } \mathrm{Re}(s) > 1).$$

It is for this reason (broadly speaking) that we say that information about the zeta function gives us information about primes. Furthermore, this is the motivation for building zeta functions over objects other than the integers. For example, the *Dedekind zeta function* is defined over an algebraic number field, K:

$$\zeta_K(s) \;=\; \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s},$$

where $\mathfrak{a}$ ranges over the non-zero ideals of the ring of integers $\mathcal{O}_K$ associated with K, and $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$. This too has an Euler product, and Dedekind [D] showed that it can be expressed as a product over only the prime ideals:

$$\zeta_K(s) \;=\; \prod_{\text{prime } \mathfrak{p} \subseteq \mathcal{O}_K} \frac{1}{1-(N(\mathfrak{p}))^{-s}} \qquad (\text{for } \mathrm{Re}(s) > 1).$$

As a final example of a zeta function with an interesting connection to our work in this paper, we mention the related zeta function over function fields. Let $\mathbb{F}$ be a finite field of $q = p^f$ elements with p prime, and let $A = \mathbb{F}[T]$ be the polynomial ring over $\mathbb{F}$. As usual, we can define a zeta function over A and find its associated Euler product for Re$(s) > 1$:

$$\zeta_A(s) \;=\; \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{1}{|f|^s} \;=\; \prod_{\substack{P \text{ irreducible} \\ P \text{ monic}}} \left(1 - \frac{1}{|P|^s}\right)^{-1},$$

where $|f|$ is defined as $q^{\deg(f)}$. However, in this setting we find the surprising fact [Ro, p. 11] that

$$\zeta_A(s) \;=\; \frac{1}{1-q^{1-s}} \quad (\text{for } \mathrm{Re}(s) > 1).$$

That is, zeta functions over function fields are rational in q^s! With these thoughts in mind, we return to juggling sequences.

4. A Norm on Juggling Sequences

Observe that when we defined zeta functions over number fields and finite fields, it was necessary to put a *norm* on the objects of interest (ideals and polynomials, respectively). Because our goal is to create a zeta function for juggling sequences, we first need to define a norm on these objects. This norm should capture as much information about the juggling sequence as possible. In the end, we opted for a norm that is both simple in form and incorporates the number of balls and the length of the juggling pattern, as follows. For a juggling sequence j, we define

$$N(j) \;=\; b^n, \tag{4.1}$$

where b is the number of balls being juggled and n is the length of the sequence. For jugglers, this has the advantage of corresponding roughly to the difficulty of the juggling pattern; increasing either the number of balls or the length of the pattern makes it more difficult to juggle. For mathematicians, we note that this does indeed satisfy the conditions of a norm:

Theorem 4.1. *Given a juggling sequence $j = (t_1, \dots, t_n)$, let as usual $b = \frac{1}{n}\sum_i^n t_i$. Then the function $N(j) = b^n$ is positive definite and multiplicative.*

Proof. The fact that the values of this function are always positive holds trivially, so we turn to multiplicativity. Let $j_1 = (t_1, \dots, t_n)$ and $j_2 = (u_1, \dots, u_m)$ be two b-ball juggling sequences, so $N(j_1) = b^n$ and $N(j_2) = b^m$. If j_1 and j_2 have the same state (a necessary condition for concatenation), then $j_1 j_2 = (t_1, \dots, t_n, u_1, \dots, u_m)$ is a sequence of length $m+n$, and therefore $N(j_1 j_2) = b^{m+n}$, as desired. □

For any subset $S \subseteq J$, we may then define the *norm counting function* for S by $\pi_S(x) = \#\{N(j) \le x \mid j \in S\}$, with $\pi(x) = \pi_{J_P}(x)$ denoting the primitive norm counting function. Note that when $b \ge 2$ one could replace the inequality $N(j) \le x$ with $n \le \frac{\log x}{\log b}$, where n is the length of the juggling sequence.

5. Zeta functions for ground state juggling sequences

Having defined a norm on juggling sequences, we can now define the zeta function for the set of ground state juggling sequences J:

$$\zeta_J(s) \;=\; \sum_{j \in J} \frac{1}{N(j)^s},$$

where as usual $s = \sigma + it$ is a complex variable, and $N(j)$ is as defined in (4.1). Having defined our zeta function, we now have three goals:

1. Rewrite the zeta function in a simpler form
2. Analytically (or at least meromorphically) continue it
3. Find its zeroes and singularities.

We will make considerable use of the following theorem concerning the number of juggling sequences of a given norm (the notation has been modified from the original version).

Theorem 5.2. [CG, Theorem 1] *Let $J(b)$ be the set of all ground state juggling sequences with b balls, and for any $j \in J$ of length n, let $N(j) = b^n$ be the norm of j. Then*

$$\#\{j \in J(b) \mid N(j) = b^n\} \;=\; \begin{cases} n! & \text{if } n < b \\ b!(b+1)^{n-b} & \text{otherwise.} \end{cases}$$

Before tackling our three goals for an arbitrary number of balls, we start with a warm-up problem, in which we examine a zeta function for the set of three-ball juggling sequences.

6. A 3-ball juggling zeta function

Recall that our zeta function in general is written as

$$\zeta_J(s) \ = \ \sum_{j \in J} \frac{1}{N(j)^s},$$

where $s = \sigma + it$ is a complex variable. We now prove several theorems about this function.

Theorem 6.3. *The 3-ball zeta function, $\zeta_J(s)$, can be written in closed form as a rational function of 3^s. In particular,*

$$\zeta_J(s) \ = \ \frac{1}{2}\left(\frac{5}{4} \cdot \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{3}{16} \cdot \frac{4}{3^s - 4}\right).$$

Proof. For the 3-ball zeta function, we take $b = 3$ in Theorem 5.2, allowing us to write

$$\begin{aligned} \zeta_J(s) &= \frac{1}{3^s} + \frac{2}{3^{2s}} + \sum_{n=3}^{\infty} \frac{\frac{3}{32} \cdot 4^n}{3^{ns}} \\ &= \frac{1}{3^s} + \frac{2}{3^{2s}} - \frac{\frac{3}{8}}{3^s} - \frac{\frac{3}{2}}{3^{2s}} + \sum_{n=1}^{\infty} \frac{\frac{3}{32} \cdot 4^n}{3^{ns}} \\ &= \frac{5}{8} \cdot \frac{1}{3^s} + \frac{1}{2} \cdot \frac{1}{3^{2s}} + \frac{3}{32} \sum_{n=1}^{\infty} \frac{4^n}{3^{ns}} \\ &= \frac{1}{2}\left(\frac{5}{4} \cdot \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{3}{16} \cdot \frac{4}{3^s - 4}\right), \end{aligned} \tag{6.2}$$

as desired. □

We have no *a priori* reason to expect to be able to write the zeta function in closed form. Neither the classic Riemann zeta function nor the Dedekind zeta function share this property. Rather, this zeta function behaves more like zeta functions over function fields.

Note also that this expression immediately gives a meromorphic continuation of the zeta function to the entire complex plane, with singularities whenever $3^s = 4$ (viz., $s = \frac{\log 4 + 2k\pi i}{\log 3}$ for any integer k).

Finally, we can use this rational version of the 3-ball zeta function to find its zeroes.

Theorem 6.4. *Let $\zeta_J(s) = \sum_{j \in J} \frac{1}{N(j)^s}$ be the 3-ball zeta function. Then the zeroes of $\zeta_J(s)$ are precisely the values $s = -\frac{\log(\frac{1}{2}(-1+\sqrt{3})) + 2k\pi i}{\log 3}$ and $s = -\frac{\log(\frac{1}{2}(1+\sqrt{3})) + (2k+1)\pi i}{\log 3}$, for all integers k.*

Proof. Setting (6.2) to zero, we set $z = 3^{-s}$ to get

$$\frac{5}{4}z + z^2 + \frac{3}{16} \cdot \frac{4z}{1 - 4z} = 0,$$

from which it soon follows via a basic algebra exercise that

$$z(-2z^2 - 2z + 1) = 0.$$

The roots of this equation are $z = 0$ and $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}$. Since $z = 3^{-s}$ is never zero, the only zeroes of $\zeta_J(s)$ occur when $s = -\frac{\log(\frac{1}{2}(-1\pm\sqrt{3}))}{\log 3}$. A simple application of complex logarithms gives the desired result. □

7. A juggling zeta function on b balls

Next, we generalize the above results to juggling zeta functions for an arbitrary number of balls.

Theorem 7.5. *For an arbitrary integer $b > 2$, the b-ball zeta function can be written as a rational function of b^s with rational coefficients; i.e., $\zeta_J(s) \in \mathbb{Q}(b^s)$.*

Proof. Recall that the b-ball zeta function is $\zeta_J(s) = \sum_{j\in J} \frac{1}{N(j)^s}$, where the number of juggling sequences with norm $N(j)$ can be found using Theorem 5.2. Applying this theorem with b balls, we find

$$\zeta_J(s) = \sum_{n=1}^{b-1} \frac{n!}{b^{ns}} + \sum_{n=b}^{\infty} \frac{b! \cdot (b+1)^{n-b}}{b^{ns}},$$

and our goal is to replace the infinite sum with a finite sum of rational terms in b^s. We have

$$\begin{aligned}
\zeta_J(s) &= \sum_{n=1}^{b-1} \frac{n!}{b^{ns}} - \sum_{n=1}^{b-1} \frac{b! \cdot (b+1)^{n-b}}{b^{ns}} + \sum_{n=1}^{\infty} \frac{b! \cdot (b+1)^{n-b}}{b^{ns}} \\
&= \sum_{n=1}^{b-1} \left(\frac{n! - b! \cdot (b+1)^{n-b}}{b^{ns}} \right) + \frac{b!}{(b+1)^b} \sum_{n=1}^{\infty} \left(\frac{b+1}{b^s} \right)^n \\
&= \sum_{n=1}^{b-1} \left(\frac{n! - b! \cdot (b+1)^{n-b}}{b^{ns}} \right) + \frac{b!}{(b+1)^{b-1}} \cdot \frac{1}{b^s - b - 1}, \qquad (7.3)
\end{aligned}$$

which proves the theorem. □

(Note that, unlike the three-ball case, we have not attempted to rewrite the general b-ball zeta function in closed form.)

Once again, the rational form of the zeta function immediately gives us meromorphic continuation to the complex plane; for general b there are singularities whenever $b^s = b+1$ (viz., at $s = \frac{\log(b+1)+2k\pi i}{\log b}$). It remains for us to find the zeroes. We find

Theorem 7.6. *Let $b > 2$ be an integer, and let $\zeta_J(s)$ be the juggling zeta function on b balls. Then the zeroes of $\zeta_J(s)$ are precisely the roots of the $(b-1)$-degree polynomial*

$$P_b(z) = 1 + \sum_{n=1}^{b-1} n!(n-b)z^n,$$

where $z = b^{-s}$.

Proof. Taking the rational form of $\zeta_J(s)$ in (7.3), we let $z = b^{-s}$ and set the expression to zero to get

$$\sum_{n=1}^{b-1}(n! - b! \cdot (b+1)^{n-b})z^n \;+\; \frac{b!}{(b+1)^{b-1}} \cdot \frac{z}{1-(b+1)z} \quad = \quad 0.$$

Taking $c_n = n! - b! \cdot (b+1)^{n-b}$ to simplify notation, this becomes

$$0 \quad = \quad \sum_{n=1}^{b-1} c_n z^n \;+\; \frac{b!}{(b+1)^{b-1}} \cdot \frac{z}{1-(b+1)z}.$$

Since $z = b^{-s}$ is never zero, multiplying first by $(1-(b+1)z)/z$, we have

$$\begin{aligned}
0 \quad &= \quad (1-(b+1)z)\sum_{n=1}^{b-1} c_n z^{n-1} \;+\; \frac{b!}{(b+1)^{b-1}} \\
&= \quad \sum_{n=1}^{b-1} c_n z^{n-1} - (b+1)\sum_{n=1}^{b-1} c_n z^n \;+\; 1 - c_1 \\
&= \quad 1 \;+\; \sum_{n=2}^{b-1} c_n z^{n-1} - (b+1)\sum_{n=1}^{b-1} c_n z^n \\
&= \quad 1 \;+\; \sum_{n=1}^{b-2} c_{n+1} z^n - (b+1)\sum_{n=1}^{b-1} c_n z^n \\
&= \quad 1 \;+\; \sum_{n=1}^{b-2} \big(c_{n+1} - (b+1)c_n\big) z^n \;-\; (b+1)c_{b-1}z^{b-1}. \qquad (7.4)
\end{aligned}$$

We can expand c_n and c_{n+1} to rewrite the coefficients of z^n as follows:

$$\begin{aligned}
c_{n+1} - (b+1)c_n \quad &= \quad (n+1)! - b! \cdot (b+1)^{n+1-b} - (b+1)(n! - b! \cdot (b+1)^{n-b}) \\
&= \quad (n+1)! - (b+1)n! + b! \cdot (b+1)^{n+1-b} - b! \cdot (b+1)^{n+1-b} \\
&= \quad (n+1)! - (b+1)n! \\
&= \quad n! \cdot (n+1-(b+1)) \\
&= \quad n! \cdot (n-b).
\end{aligned}$$

Substituting this into (7.4) we find that the zeroes of the zeta function are given by the roots of the $(b-1)$-degree polynomial

$$\begin{aligned}
P_b(z) &= 1 + \sum_{n=1}^{b-2} n!(n-b)z^n - (b+1)((b-1)! - b!(b+1)^{-1})z^{b-1} \\
&= 1 + \sum_{n=1}^{b-2} n!(n-b)z^n - (b-1)!((b+1)-b)z^{b-1} \\
&= 1 + \sum_{n=1}^{b-2} n!(n-b)z^n - (b-1)! \cdot z^{b-1} \\
&= 1 + \sum_{n=1}^{b-1} n!(n-b)z^n,
\end{aligned}$$

as desired. □

(It should be noted that a similar polynomial appears in a pair of generating functions in [CG, pp. 189-190].)

Each root r of $P_b(z)$ will produce an infinite family of zeroes: $s = -\frac{\operatorname{Log} r + 2k\pi i}{\log b}$, where k is any integer and $\operatorname{Log} r$ denotes the principal value of the logarithm. Clearly, the zeroes will be distributed more tightly (in the vertical direction) for larger values of b.

8. Locating the zeroes of the zeta function on b balls

We continue the analysis of the various $\zeta_J(s)$ with a more precise accounting of their zeroes. For example, in the case $b = 4$ the roots of $P_4(z) = 1 - 3z - 4z^2 - 6z^3$ are

$$0.23435808..., \quad -0.45051237... \pm i \cdot 0.71288220... .$$

Since each of these roots has a modulus ≤ 1, and since $z = 4^{-s}$, it follows that each zero s_0 of $\zeta_J(s)$ satisfies $\text{Re}(s) > 0$. However, it is possible to extend this result further by using the following

Theorem 8.7. *For any $b \geq 4$, the zeroes s_0 of $\zeta_J(s)$ satisfy $Re(s_0) > 0$.*

Proof. The case $b = 4$ has already been attended to above. For $b \geq 5$, we will make use of the inequality

$$\sum_{n=1}^{b-2} n!(b-n) < (b-1)! - 1 \qquad (b \geq 5) \tag{8.5}$$

which can be shown by induction. Next, for any $b \geq 5$ we define the polynomials

$$f_b(z) = 1 - (b-1)!z^{b-1}, \qquad g_b(z) = \sum_{n=1}^{b-2} n!(n-b)z^n .$$

Note that $f_b(z)$ comprises the first and last terms of $P_b(z)$, while $g_b(z)$ is made up of the remaining terms. Taking C to be the unit circle (i.e., $C = \{z = e^{2\pi it} \mid 0 \leq t < 1\}$), we have by (8.5)

$$\left|g_b(z)\right| \;\le\; \sum_{n=1}^{b-2} n!(b-n) \;<\; (b-1)!-1 \;\le\; \left|f_b(z)\right|$$

on C. Since $P_b(z) = f_b(z) + g_b(z)$, Rouché's theorem implies that, inside the unit circle, the polynomial $P_b(z)$ has the same number of roots as $f_b(z)$. Since *all* the $b-1$ roots of $f_b(z)$ have the modulus $1/\sqrt[b-1]{(b-1)!} < 1$, we have proven the theorem. □

We have thus far defined the zeroes of $\zeta_J(s)$ implicitly as the roots of a polynomial. In this section we go further, and give upper and lower bounds on the (real part) of the zeroes, locating them more explicitly on the complex plane. To do so, we require the following inequalities.

Lemma 8.1. *Let b denote a positive integer. Then we have*

$$\frac{b^{(b-1)/2}}{(b-4)!} \quad < \quad 1 \qquad (b \ge 18)\,, \tag{8.6}$$

$$\frac{n!(b-n)}{b^{n/2}} \quad \le \quad \frac{4(b-4)!}{b^{(b-4)/2}} \qquad (b \ge 10\,,\ 1 \le n \le b-4)\,. \tag{8.7}$$

Lemma 8.2. *Let b denote a positive integer. Then we have*

$$\sum_{n=1}^{b-2} \frac{n!(b-n)}{b^{n/2}} \;<\; \frac{(b-1)!}{b^{(b-1)/2}} - 1 \qquad (b \ge 16)\,. \tag{8.8}$$

Proofs of these inequalities are appended to the end of this section. Here, we use these facts to establish a stronger result regarding the zeroes of $\zeta_J(s)$:

Theorem 8.8. *For $b \ge 16$, the zeroes s_0 of $\zeta_J(s)$ satisfy the lower bound $Re(s_0) > \frac{1}{2}$. For $b \ge 2$, the zeroes of $\zeta_J(s)$ satisfy the upper bound $Re(s_0) < \frac{\log(b+1)}{\log b}$.*

Proof. Repeating the above proof with the curve C defined by $\{z = e^{2\pi i t}/\sqrt{b} \mid 0 \le t < 1\}$, we use (8.8) to conclude that

$$\left|g_b(z)\right| \le \sum_{n=1}^{b-2} n!(b-n)\frac{1}{b^{n/2}} \;<\; \frac{(b-1)!}{b^{(b-1)/2}} - 1 \;\le\; \left|f_b(z)\right|.$$

In this situation Rouché's theorem and the inequality $1/\sqrt[b-1]{(b-1)!} \;<\; 1/\sqrt{b}$ $(b \ge 6)$ show that for $b \ge 16$ the roots z_i of $P_b(z)$ satisfy $|z_i| < 1/\sqrt{b}$ $(i = 1, \ldots, b-1)$. Since $z = b^{-s}$, we estimate below the real part of every zero s_0 of $\zeta_J(s)$ by the modulus of the corresponding root $z_0 \in \{z_1, \ldots, z_{b-1}\}$ of $P_b(z)$:

$$\mathrm{Re}(s_0) \;=\; -\frac{\log|z_0|}{\log b} \;=\; \frac{\log(1/|z_0|)}{\log b} \;>\; \frac{\log\sqrt{b}}{\log b} \;=\; \frac{1}{2} \qquad (b \ge 16)\,.$$

For an upper bound of $\mathrm{Re}(s_0)$ we use the identity

$$\sum_{n=1}^{b-1} n!(b-n)\frac{1}{(b+1)^n} \;=\; 1-\frac{(b+1)!}{(b+1)^b} \qquad (b\geq 2)\,,$$

which shows that

$$\left|P_b(z)\right| \;\geq\; 1-\sum_{n=1}^{b-1} n!(b-n)\frac{1}{(b+1)^n} \;=\; \frac{(b+1)!}{(b+1)^b} \;>\; 0$$

holds for $|z|\leq \frac{1}{b+1}$. Consequently, the zeroes z_0 of $P_b(z)$ satisfy $|z_0|>\frac{1}{b+1}$, and therefore $\mathrm{Re}(s_0)$ satisfies

$$\mathrm{Re}(s_0) \;<\; \frac{\log(b+1)}{\log b} \qquad (b\geq 2)$$

for every zero s_0 of $\zeta_J(s)$. This completes the theorem. □

Remark 8.1. The upper bound for $\mathrm{Re}(s_0)$ in Theorem 8.8 cannot be replaced by 1: since

$$P_b\Big(\frac{1}{b+1}\Big) = \frac{(b+1)!}{(b+1)^b} > 0 \qquad \text{and} \qquad P_b\Big(\frac{1}{b}\Big) \leq -\frac{1}{b}-\frac{2}{b^2}+\frac{18}{b^3} < 0$$

hold simultaneously for $b\geq 4$, there is a real zero s_0 of $\zeta_J(s)$ between 1 and $\frac{\log(b+1)}{\log b}$ for such b.

Given $b\geq 16$, we will call the region $\mathcal{S}=\{z\in\mathbb{C}\mid \frac{1}{2}<\mathrm{Re}(z)<\frac{\log(b+1)}{\log b}\}$ the *critical strip* for $\zeta_J(s)$. Next, for any positive real number X we may define the rectangle $\mathcal{S}_X$ in the critical strip by $\frac{1}{2}\leq \mathrm{Re}(s)\leq \frac{\log(b+1)}{\log b}$ and $-X<\mathrm{Im}\,(s)<X$. Since $\zeta_J(s)$ is periodic in $\mathrm{Im}(s)$ with period $\frac{2\pi}{\log b}$, and since $P_b(z)$ has degree $b-1$, there are exactly $b-1$ zeroes (up to multiplicity) in $\mathcal{S}_{2\pi/\log b}$. This discussion proves the following

Theorem 8.9. *For any $b\geq 16$, the number $N_b(\zeta_J)$ of zeroes in a general $\mathcal{S}_X$ satisfies*

$$N_b(\zeta_J) \;=\; \frac{X(b-1)\log b}{\pi}+\mathcal{O}_b(1).$$

The implicit constant of the error term depends at most on b.

Collecting together our results on the singularities of $\zeta_J(s)$ (stated before in the comments preceding Theorem 7.6) and on the zeroes of $\zeta_J(s)$ (stated in Theorem 8.9), we have the following corollary.

Corollary 8.1. *Let C be the boundary of the rectangle in the complex plane defined by the four points $1/2-iX$, $2-iX$, $2+iX$, and $1/2+iX$, where X is any positive real number such that the function $\zeta_J'(s)/\zeta_J(s)$ with $b\geq 16$ is holomorphic on C. Let $S_b(\zeta_J)$ denote the number of singularities of $\zeta_J(s)$ in the rectangle bounded by C. Then we have*

$$N_b(\zeta_J)-S_b(\zeta_J) \;=\; \frac{1}{2\pi i}\oint_C \frac{\zeta_J'(s)}{\zeta_J(s)}\,ds \;=\; \frac{X(b-2)\log b}{\pi}+\mathcal{O}_b(1)\,.$$

The implicit constant of the error term depends at most on b.

Proof of Lemma 8.1: For (8.6) we proceed by induction. For $b = 18$ we have

$$\frac{18^{17/2}}{14!} = 0.5363001782... < 1\,.$$

Next, let (8.6) be true for some integer $b \geq 18$. Then,

$$\begin{aligned}\frac{(b+1)^{b/2}}{(b-3)!} &= \frac{b^{1/2}(1+1/b)^{b/2}}{(b-3)} \cdot \frac{b^{(b-1)/2}}{(b-4)!} \\ &< \frac{b^{1/2}(1+1/b)^{b/2}}{b-3} \qquad \text{(Induction hypothesis)} \\ &< \frac{\sqrt{be}}{b-3} < 1 \qquad (\text{with } e := \exp(1))\,.\end{aligned}$$

To prove (8.7) we first check the inequality for $b = 10, \ldots, 44$ by computer. Then we assume that $b \geq 45$ and distinguish the cases $n \in \{b-4, b-5, b-6\}$ and $n < b-6$. For $n = b-4$ there is nothing to show because (8.7) becomes an identity. For $n = b-5$ the inequality (8.7) is equivalent to

$$\frac{5}{4} \leq \frac{b-4}{\sqrt{b}}\,,$$

and for $n = b-6$ it is equivalent to

$$\frac{3}{2} \leq \frac{(b-4)(b-5)}{b}\,.$$

Both inequalities hold for $b \geq 45$. Now let $b \geq 45$ and $1 \leq n \leq b-7$. From Stirling's formula,

$$\sqrt{2\pi m}\Big(\frac{m}{e}\Big)^m < m! < \sqrt{2\pi(m+1)}\Big(\frac{m}{e}\Big)^m \qquad (m \geq 1)\,,$$

we obtain on the one side

$$\begin{aligned}\frac{n!(b-n)}{b^{n/2}} &< \frac{n!b}{b^{n/2}} < \frac{b\sqrt{2\pi(n+1)}}{b^{n/2}} \cdot \Big(\frac{n}{e}\Big)^n \\ &= b\sqrt{2\pi(n+1)}\Big(\frac{n}{e\sqrt{b}}\Big)^n \\ &\leq b\sqrt{2\pi(b-6)}\Big(\frac{b-7}{e\sqrt{b}}\Big)^n \\ &< b\sqrt{2\pi(b-4)}\Big(\frac{b-7}{e\sqrt{b}}\Big)^{b-7}\,.\end{aligned} \tag{8.9}$$

Note that $b - 7 > e\sqrt{b}$ holds for $b \geq 45$. On the other side, we have

$$\frac{4(b-4)!}{b^{(b-4)/2}} > \frac{4\sqrt{2\pi(b-4)}}{b^{(b-4)/2}} \cdot \Big(\frac{b-4}{e}\Big)^{b-4} = 4\sqrt{2\pi(b-4)} \cdot \Big(\frac{b-4}{e\sqrt{b}}\Big)^{b-4}\,. \tag{8.10}$$

For $b \geq 45$ we obtain the following inequalities (the first one does not hold for $b = 44$):

$$\begin{aligned} & e^3 b^{5/2} \;<\; 4(b-4)^3 \\ \iff\quad & e^3 b^{5/2}(b-4)^{b-7} \;<\; 4(b-4)^{b-4} \\ \Longrightarrow\quad & e^3 b^{5/2}(b-7)^{b-7} \;<\; 4(b-4)^{b-4} \\ \iff\quad & \frac{b(b-7)^{b-7}}{(e\sqrt{b})^{b-7}} \;<\; 4\cdot\frac{(b-4)^{b-4}}{(e\sqrt{b})^{b-4}} \\ \iff\quad & b\sqrt{2\pi(b-4)}\Big(\frac{b-7}{e\sqrt{b}}\Big)^{b-7} \;<\; 4\sqrt{2\pi(b-4)}\cdot\Big(\frac{b-4}{e\sqrt{b}}\Big)^{b-4}. \end{aligned} \tag{8.11}$$

Then, (8.7) for $b \geq 45$ and $1 \leq n \leq b-7$ follows from (8.9), (8.10), and (8.11). This completes the proof of the lemma. □

Proof of Lemma 8.2: First of all, note that the left side may be written as $g_b(\frac{1}{\sqrt{b}})$. To verify (8.8), one first checks by computer that it holds for $b = 16, \dots, 44$, but not for $b = 15$. Next, let $b \geq 45$. The polynomial $p(x) := x^6 - 6x^5 - 9x^4 + 26x^3 + 20x^2 - 12x - 7$ has six real roots $x_1, \dots, x_6$ with $-2 < x_1 < \cdots < x_6 < 6.705$. Therefore, for $x > 6.705$ we obtain $p(x) > 0$, and hence, equivalently,

$$4(x^2-4)x^3 + 3(x^2-3)x^2 + 2(x^2-3)(x^2-2)x \;<\; (x^2-3)(x^2-2)(x^2-1) - 1\,.$$

Substituting $b := x^2$, we have for integers $b \geq 45$ that

$$\begin{aligned} & 4(b-4)b^{3/2} + 3(b-3)b + 2(b-3)(b-2)b^{1/2} \\ <\quad & (b-3)(b-2)(b-1) - 1 \\ <\quad & (b-3)(b-2)(b-1) - \frac{b^{(b-1)/2}}{(b-4)!} \qquad \text{(by (8.6))}\,. \end{aligned}$$

Multiplying with $(b-4)!/b^{(b-1)/2}$, we arrive at

$$(b-4)\frac{4(b-4)!}{b^{(b-4)/2}} + \frac{3(b-3)!}{b^{(b-3)/2}} + \frac{2(b-2)!}{b^{(b-2)/2}} \;<\; \frac{(b-1)!}{b^{(b-1)/2}} - 1\,.$$

Now we apply (8.7) to estimate below the first term on the left-hand side:

$$\sum_{n=1}^{b-4}\frac{n!(b-n)}{b^{n/2}} + \frac{3(b-3)!}{b^{(b-3)/2}} + \frac{2(b-2)!}{b^{(b-2)/2}} \;<\; \frac{(b-1)!}{b^{(b-1)/2}} - 1\,.$$

This is (8.8) for $b \geq 45$. □

9. Conclusion

We have not made use of an Euler product for our zeta function, such as

$$\zeta(s) \;=\; \prod_{j\in J_P}\big(1 - N(j)^{-s}\big)^{-1},$$

in this work. It would be interesting to develop another version of meromorphic continuation using this formula. Two basic methods suggest themselves. The first is to expand the product into a Dirichlet series (this would differ significantly from the series used above) and then attempt a meromorphic continuation using methods similar to Riemann's. The other method would be to calculate the logarithmic derivative of $\zeta(s)$ and then attempt a meromorphic continuation of that. In each case, the multiplicity is the salient feature; we need a count for the number of primitive juggling sequences of a given norm.

An asymptotic count of this type appears in [CG, p. 191]. We also find a claim by Benoit Cloitre in the discussion for sequence A084519 of the OEIS [S] that

$$\#\{j \in J_P \mid N(j) = 3^n\} \;\sim\; \alpha \cdot \beta^n,$$

where $\beta \approx 3.6891$ is the real root of the polynomial $p(t) = t^3 - 3t^2 - 2t - 2$ and $\alpha \approx 0.068706$ is the real root of the polynomial $q(t) = 118t^3 + 118t^2 + 35t - 3$. In fact, it is conjectured that the multiplicity may be calculated exactly by rounding $\alpha \cdot \beta^n$ to the nearest integer. Perhaps some version of this can be used to develop an Euler product for the juggling zeta function.

Concerning the zeroes of $\zeta_J(s)$, the contour integral in Corollary 8.1 can be easily computed numerically for a given b and X by using a suitable computer algebra system and by splitting the path of integration into the four edges of the rectangle. By numerical investigations using such a computer program it seems reasonable that the error term $\mathcal{O}_b(1)$ in Corollary 8.1 becomes very small in general.

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