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The nonlinear progressive wave equation (NPE) [McDonald and Kuperman, J. Acoust. Soc. Am. **81**, 1406–1417 (1987)] is expressed in a form to accommodate changes in the ambient atmospheric density, pressure, and sound speed as the time-stepping computational window moves along a path possibly traversing significant altitude differences (in pressure scale heights). The modification is accomplished by the addition of a stratification term related to that derived in the 1970s for linear range-stepping calculations and later adopted into Khokhlov-Zabolotskaya-Kuznetsov-type nonlinear models. The modified NPE is shown to preserve acoustic energy in a ray tube and yields analytic similarity solutions for vertically propagating N waves in isothermal and thermally stratified atmospheres.
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I. INTRODUCTION

Recent interest in engineering supersonic jets with minimal sonic boom signatures[1,2] has motivated research in improved modeling capability for propagation of weakly nonlinear waves from a high altitude source down to ground level. One envisions carrying out an ensemble of calculations for various seasonal atmospheric profiles and turbulence distributions. The desire for computational efficiency motivates the use of a wave-following computational window so that the entire horizontal and vertical extent from source to ground stations need not be updated at every step.

In this paper, we present a modified version of the nonlinear progressive wave equation (NPE) model[3] for nonlinear wave propagation in stratified atmospheres. The original time-marching NPE formulation was developed for the ocean where the ambient density is nearly constant. We adapt a correction term originally derived for range-marching linear propagation calculations[4,5] and later used in nonlinear calculations[6–9] to facilitate accurate wave propagation through realistically stratified atmospheres. The current work may be extended as in previous work[10] to include effects of turbulence and relaxation processes.

II. THEORY

The NPE describes the evolution of finite amplitude acoustic density fluctuations ρ' in a wave-following coordinate system (Fig. 1) moving in the x direction at a nominal sound speed c_0, where subscript zero refers to the undisturbed medium ahead of the wave. NPE is a time-marching model as opposed to the range-marching Khokhlov-Zabolotskaya-Kuznetsov (KZK) model.[11] Time marching formulation allows for compatibility with fluid dynamic models and is helpful in visualizing wave evolution. NPE and KZK are equally valid within their respective assumptions.[12] Whereas range-marching has been widely used in beam propagation,[4,5,8,11] the time domain formulation of NPE is well suited to impulsive sources. One of the motivations for casting NPE in the time domain was to allow the direct introduction of shock capturing methods developed in computational fluid dynamics such as was done in Ref. 3.

The NPE for unstratified media is

$$D_t\rho' = -\frac{1}{2c_0}\partial_x\left[p'(\rho') + c_0^2\left(\rho'^2/\rho_0 - \rho'\right)\right] + \frac{c_0}{2}\int_x^\infty \nabla_\perp^2\rho' dx, \tag{1}$$

where $D_t = \partial_t + c_0\partial_x$ is the time derivative in the moving frame, ρ_0 is the unperturbed density of the medium with $\rho = \rho_0 + \rho'$, p' is acoustic overpressure, and $\nabla_\perp^2 \equiv (\partial_y^2 + \partial_z^2)$. The x–integration path in Eq. (1) terminates in the quiescent medium ahead of the wave where ρ' and its derivatives are zero. Error terms in Eq. (1) are $O(\rho'^3, \rho'^2\theta^2)$, where θ is the wavenormal angle with respect to x. For this reason, it is sufficient to use an adiabatic equation of state $p'(\rho')$ because weak shock heating is cubic in shock amplitude.[13] For an adiabatic gas, Eq. (1) becomes to the same order approximation

$$D_t\rho' = -\partial_x\left(c_1\rho' + \beta c_0\frac{\rho'^2}{2\rho_0}\right) + \frac{c_0}{2}\int_x^\infty \nabla_\perp^2\rho' dx, \tag{2}$$

where $c_1 = c(\mathbf{r}) - c_0$, $\beta = (\gamma+1)/2$, and γ is the ratio of specific heats ($\gamma = 1.4$ for air, so $\beta = 1.2$).

A. Correction for density stratification

The right side of Eq. (2) describes weak but cumulative physical processes: refraction, nonlinear steepening, and combined geometric spreading and diffraction in the

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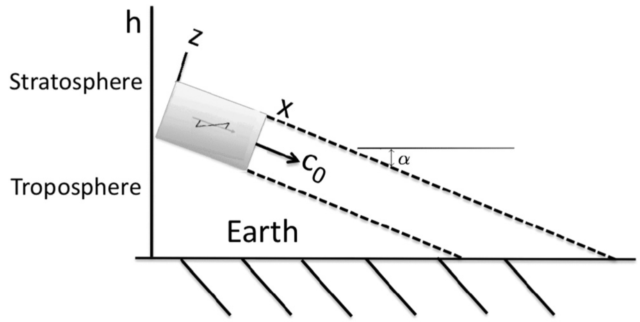

FIG. 1. The NPE computational window moves at the local sound speed c_0 through a stratified atmosphere at an arbitrary angle α to the horizontal. α is taken positive for upward propagation and negative for downward.

transverse Laplacian. For wavelengths much less than a scale height, the stratification effect will also be weak and cumulative. From Eq. (9) of Ref. 6, the stratification term (absent all other processes) for the acoustic pressure in a Cartesian coordinate system would be

$$\frac{\partial p'}{\partial s} = \frac{p'}{2}\frac{d}{ds}\ln(\rho_0 c_0) + \cdots \tag{3}$$

where s is distance along the propagation direction, and ρ_0 and c_0 are the altitude dependent density and sound speed profiles. In the time- marching notation of the NPE, Eq. (3) would be

$$c_0^{-1} D_t p' = \frac{p'}{2}\frac{d}{dx}\ln(\rho_0 c_0) + \cdots. \tag{4}$$

Note that the range derivatives in Eq. (3) are represented in Eq. (4) by a time derivative on the left and a range derivative on the right. (The Appendix shows from the stratified linear wave equation that this is the proper substitution.) One can derive from Eq. (4) (or from the stratified wave equation as in the Appendix) a correction term for the dimensionless acoustic overdensity, defined as

$$\tilde{\rho}' \equiv \frac{\rho'(\mathbf{r}, t)}{\rho_0(h_0)} \tag{5}$$

where h_0 is altitude at the center of the moving computational window. The stratification term from Eq. (A7) is appended to Eq. (2), resulting in

$$D_t\tilde{\rho}' = -\partial_x\left(c_1\tilde{\rho}' + \frac{\beta c_0}{2}\tilde{\rho}'^2\right) + \frac{c_0}{2}\int_x^{\infty}\nabla_\perp^2\tilde{\rho}' dx - \frac{1}{2}c_0\tilde{\rho}'\partial_x\ln(p_0 c_0), \tag{6}$$

where ambient quantities and their derivatives are evaluated at the center of the computational window. Equation (6) is the NPE including atmospheric stratification. Note in Eq. (6) that the stratification term for the overdensity involves $[-\ln(\rho_0 c_0)]$, while the stratification term for the pressure involves $[+\ln(\rho_0 c_0)]$. This illustrates that care must be used when transplanting terms from one nonlinear acoustic formulation to another.

III. ENERGY CONSERVATION IN A RAY TUBE

We will demonstrate that Eq. (6) retains the physics that ensures energy conservation within a ray tube (i.e., the simplest form[14] of the Blokhintsev invariant[15] of ray acoustics). In a static three dimensional atmosphere (no winds), the Blokhintsev invariant reduces to

$$\frac{p'^2 A}{\rho_0 c_0} = \text{const}, \quad \text{or} \quad \rho'^2 A p_0 c_0 = \text{const}, \tag{7}$$

where A is the cross sectional area of the ray tube (away from caustics).

We assume that level surfaces (surfaces of constant $\tilde{\rho}'$) are smooth, with radii of curvature much larger than wavelengths of interest (in order to make rays a valid description of the acoustic field). Let O designate the point where a ray of interest passes through a given level surface.

Rays are normal to level surfaces, and x is taken to be parallel to the ray passing through point O. In the local coordinate system near O, we have (see the Appendix)

$$\nabla_\perp^2\tilde{\rho}' = \left(r_1^{-1} + r_2^{-1}\right)\partial_x\tilde{\rho}' \tag{8}$$

where r_1 and r_2 are the principal radii of curvature of the level surface at O. The sign of $r_{1,2}$ is taken to be positive for convex (expanding) curvature, and negative for concave (collapsing) curvature. The result [Eq. (8)] is derived from basic principles in Eq. (A10) and alternatively is obtained in an orthogonal curvilinear coordinate system (x, ϕ_1, ϕ_2) with transverse variables lying on level surfaces. The angle variables $\phi_{1,2}$ are the angles between the x axis and the surface tangents parallel to the orthogonal principal directions near point O. The increment of arc length is given by $ds^2 = dx^2 + r_1^2 d\phi_1^2 + r_2^2 d\phi_2^2$. Then a result of differential geometry[16] and local symmetry with respect to transverse coordinates near O leads to Eq. (8).

For the purposes of this section, we augment the moving time derivative to include the nonlinear increment in sound speed: $D_t \to \partial_t + c_0(1 + \beta\tilde{\rho}')\partial_x$. In ray acoustics, surfaces of constant phase advance with the local sound speed $c = c_0(1 + \beta\tilde{\rho}')$. The level surfaces discussed in the preceding text coincide with constant phase surfaces within the assumptions of ray acoustics. A ray tube of small but arbitrary cross sectional area A may be partitioned into infinitesimal rectangular elements $\delta A = \delta l_1 \cdot \delta l_2$ aligned with the principal directions of the level surface. Elementary geometry shows that each side $(\delta l_1, \delta l_2)$ of a rectangular element expands (or contracts) as $\partial_t(\delta l_1, \delta l_2) = \left(c r_1^{-1}\delta l_1, c r_2^{-1}\delta l_2\right)$, with the result $D_t\delta A = \delta A c\left(r_1^{-1} + r_2^{-1}\right)$. Thus $D_t A = Ac\left(r_1^{-1} + r_2^{-1}\right)$, or

$$D_t \ln A = c\cdot\left(r_1^{-1} + r_2^{-1}\right). \tag{9}$$

Using Eqs. (8) and (9), we may carry out the integral in Eq. (6) by parts, resulting in

$$D_t\tilde{\rho}' = -\partial_x(c_1\tilde{\rho}') - \frac{\tilde{\rho}'}{2}D_t\ln A - \frac{\tilde{\rho}'}{2}D_t\ln(p_0 c_0) + O\left(r_{1,2}^{-2}\tilde{\rho}'\right). \tag{10}$$

The last term is dropped as being much smaller than the other terms away from caustics. If the ray tube is taken to have infinitesimal cross section, then c_1 in Eq. (10) approaches zero within the ray tube cross section, and Eq. (10) becomes

$$D_t\left(\tilde{\rho}'^2 A p_0 c_0\right) = 0, \tag{11}$$

which according to Eq. (7) preserves the ray tube energy. Equation (11) is equivalent to an expression for acoustic energy in a ray tube derived in the high frequency limit of nonlinear geometric acoustics.[14] For a ray tube oriented at a small angle θ to the x direction, the error in Eq. (11) is $O(\tilde{\rho}'^2\theta^2)$. It should be noted that the unstratified NPE theory preserves total acoustic energy without reference to ray tubes [Ref. 3, Eq. (27)].

IV. STRATIFICATION EFFECTS ON N WAVES

A. Self-similar shock profiles

The effect of the stratification term in Eq. (6) on the development of N waves may be investigated using the following one dimensional version of Eq. (6):

$$D_t\tilde{\rho}' = -\beta c_0\tilde{\rho}'\partial_x\tilde{\rho}' - c_0\tilde{\rho}' \cdot \frac{1}{2}\partial_x \ln(p_0c_0). \tag{12}$$

Equation (1), which was derived[3] in the form of Eq. (2) from the Euler equations, possesses a subset of similarity solutions[17] $\rho'(\mathbf{r},t) = \rho'(\mathbf{r}/t)$ for propagation in homogeneous media. Nonlinear waves in the absence of dissipation tend to age toward stable self-similar forms.[18] A modification of the similarity variable is required for Eq. (12) because of the stratification term. Define

$$\delta \equiv \frac{1}{2}\partial_x \ln(p_0c_0), \tag{13}$$

and let x be oriented vertically upward or downward and consider δ and c_0 constant as in an isothermal atmosphere. In a more realistic atmosphere, δ and c_0 both vary slowly as compared to p_0 or ρ_0. (We will show in the following text that the product $c_0\delta$ is very nearly constant for a standard atmospheric model.)

The similarity variable appropriate for Eq. (12) is not x/t but rather x/u, where

$$u = \left(e^{c_0t\delta} - 1\right)/(c_0\delta), \quad t > 0. \tag{14}$$

The similarity solution of Eq. (12) is now

$$\tilde{\rho}'(x,t) = \frac{x}{\beta c_0 u} = \frac{x\delta}{\beta(e^{c_0t\delta} - 1)}, \tag{15}$$

which is verified by substitution into Eq. (12), recalling that x is taken relative to the origin of the moving frame. We take shock discontinuities to be at $x = \pm x_s$, so that from Eq. (12) and the Rankine–Hugoniot condition for mass conservation,[15] we have

$$D_t x_s = \frac{1}{2}\beta c_0\tilde{\rho}'(x_s,t) \quad \text{and}$$
$$x_s = x_0\sqrt{|1 - e^{-c_0t\delta}|} \tag{16}$$

where x_0 is a constant. In the limit $\delta \to 0$, one finds $u \to t$, and the proper forms for homogeneous media are recovered:

$$\tilde{\rho}' = x/(\beta c_0 t), |x| < x_s \quad \text{and} \quad x_s \propto \sqrt{t}. \tag{17}$$

Results from Eqs. (15) to (17) are shown in Fig. 2, converted to pressure by the linear relation $p' = \tilde{\rho}'\rho_0c_0^2 = \gamma p_0\tilde{\rho}'$, where ρ_0,c_0, and p_0 depend appropriately on altitude. On the left is the unstratified result [Eq. (17)]. Equation (17) has no inherent time or length scale, but for comparison with the stratified N wave [Eqs. (15) and (16)], which does, we consider vertical propagation over a distance of 6.4 km, or about 0.8 pressure scale heights $H_p = 8$ km in a stratified isothermal atmosphere with $c_0 = 331$ m/s. We assign nominal N wave peak pressure and width 100 Pa and 40 m. All three parts of Fig. 2 begin with the same initial condition. In Fig. 2(b), we have an upward going wave, so we have $\delta = -0.0625$ km^{-1}. In Fig. 2(c), the downward going wave encounters increasing density $\delta = 0.0625$ km^{-1}.

It is evident from Fig. 2 that N wave pressure profiles propagating into a decreasing density age faster than in a nonstratified atmosphere. Conversely, N waves propagating into an increasing density age slower than in in a nonstratified atmosphere.

In the isothermal examples of Figs. 2(b) and 2(c), using the stated values $H_p = 8$ km and $c_0 = 331$ m/s, the product $|c_0\delta|$ has the constant value 0.2069 s^{-1} with a negative sign for upward propagation and positive for downward. The earth's atmosphere is not isothermal, but values for $|c_0\delta|$ computed from the ISO 2533:1975 model[19] vary only slightly: (0.02206, 0.02433, 0.02309, 0.02230) s^{-1} at altitudes (0, 11, 20, 32) km, with monotonic variation between these altitudes. For this reason, the similarity solution of Eqs. (15) and (16) with $|c_0\delta|$ constant should be descriptive of Earth's nonisothermal atmosphere.

B. N wave energetics

The energy per unit area of the propagating N wave is

$$E = \int_{-x_s}^{x_s} \frac{p'^2}{\rho_0c_0^2}dx = \int_{-x_s}^{x_s} \tilde{\rho}'^2\rho_0c_0^2dx = \gamma p_0\int_{-x_s}^{x_s} \tilde{\rho}'^2 dx. \tag{18}$$

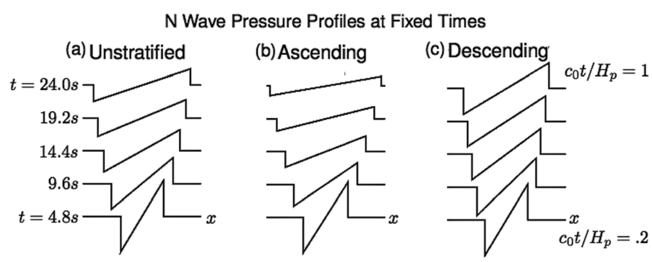

FIG. 2. Self-similar N wave development during 6.4 km propagation at 331 m/s in unstratified and isothermal atmospheres from Eqs. (14) to (17). (a) Unstratified, (b) upward propagation (pressure scale height $H_p = 8$ km), (c) downward propagation. Time is expressed on the left in seconds and on the right is converted to scale heights.

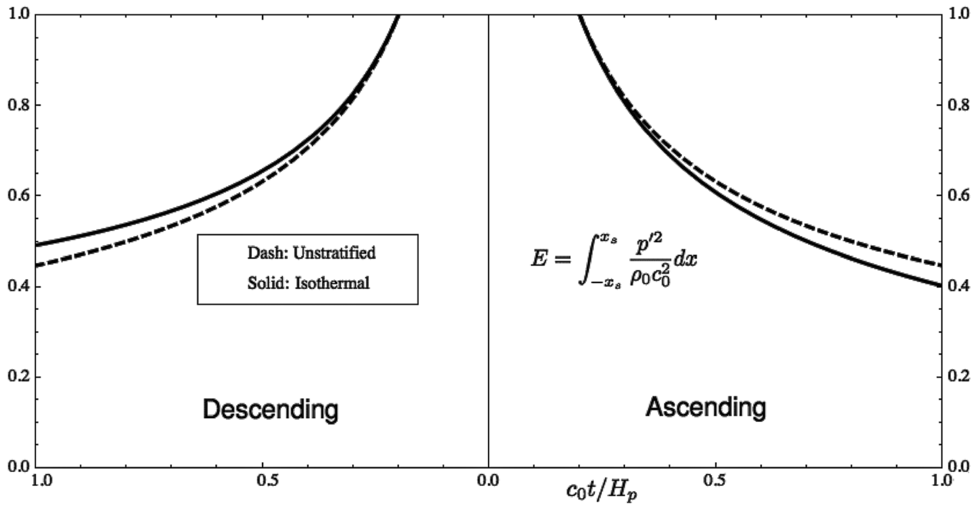

FIG. 3. Integrated energy per unit area in the N waves of Fig. 2. Dashed line: unstratified atmosphere; solid line: isothermal with scale height 8 km. Energies are normalized to the initial condition.

The last expression in Eq. (18) takes the ambient variables to be constant over the extent of the N wave. For an unstratified atmosphere, Eqs. (17) and (18) lead to

$$E \propto t^{-1/2}. \tag{19}$$

For an isothermal atmosphere, Eq. (13) implies for the coordinate frame moving vertically with the N wave, $p_0 \propto \exp(2c_0 t\delta)$ because c_0 is constant. Then Eqs. (15), (16), and (18) lead to

$$E \propto \left|1 - e^{-c_0 t\delta}\right|^{-1/2}. \tag{20}$$

The evolution of the N wave energy over the span illustrated in Fig. 2 is shown in Fig. 3 for unstratified and isothermal atmospheres. After downward propagation of 0.8 scale heights, the isothermal N wave has retained 49.2% of its initial energy, as compared to 44.7% for the unstratified N wave. For upward propagation of the same distance, the isothermal N wave has retained 40.2% of its initial energy. The change in energy for upward versus downward isothermal N waves is less than the change in pressure amplitudes revealed by Fig. 2 because of the presence of the bulk modulus in Eq. (18).

C. N wave energetics in a non- isothermal atmosphere

Taking $c_0\delta$ effectively constant as shown in the preceding text for the ISO 2533:1975 model,[19] we can evaluate Eq. (18) with c_0 not held constant. Taking the atmosphere to be adiabatic (a fair approximation in the troposphere up to about 10 km), it is straightforward to show that

$$c_0 \propto (p_0 c_0)^a, \quad a = \frac{\gamma - 1}{3\gamma - 1}. \tag{21}$$

For $\gamma = 1.4$, $a = 1/8$. The same steps that led to Eq. (20) now result in

$$E \propto e^{-c_0 t\delta/4}\left|1 - e^{-c_0 t\delta}\right|^{-1/2} = \left|2\sinh\frac{c_0 t\delta}{2}\right|^{-1/2}. \tag{22}$$

It is interesting that $\gamma = 1.4$ is the unique value that results in Eq. (22) being symmetric with respect to $c_0 t\delta$ (i.e., upward or downward propagation). When overlaid onto Fig. 3, Eq. (22) is indistinguishable from the dashed (unstratified) curve [Eq. (19)]. So the primary effect of adiabatically restoring a realistic sound speed profile to the isothermal result [Eq. (20)] is that the N wave energy loss rate for vertical propagation becomes nearly identical to that of an unstratified atmosphere.

V. SUMMARY

We have shown that the addition of a correction term involving a logarithmic derivative of the product $p_0 c_0$ to the NPE model as in Eq. (6) allows the wave-following window to traverse significant altitude differences as measured in pressure scale heights. The stratification term is closely related to that used in range-marching models[4–9] but with differences as discussed after Eq. (6). The analysis applies to wavelengths much shorter than ambient scale heights. The modified wave equation has been shown to preserve energy within a ray tube (i.e., the simplest form of the Blokhintsev invariant). It also results in a self-similar N wave analytic solution [Eq. (16)], which reverts to the proper form for homogeneous media in the limit of weak stratification. The N wave solution shows increased nonlinear aging for propagation into decreasing density and decreased aging for propagation into increasing density. The rate of N wave energy loss has been derived in Eqs. (19) to (22) for unstratified, isothermal, and adiabatically stratified atmospheres.

ACKNOWLEDGMENT

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APPENDIX: DERIVATION OF THE NPE STRATIFICATION TERM

We will now derive a one dimensional linear analog for Eq. (2) (absent refraction, nonlinearity and the transverse Laplacian term) corrected to first order in ambient gradients. The resulting stratification term will then be appended to

Eq. (2). Validation of this procedure will demonstrate (1) that the resulting equation preserves energy in a ray tube (at frequencies high enough to make ray acoustics valid) and (2) that it gives physically realistic expressions for N wave profiles and their energetics. This type of stratification term has been previously derived[4,5] for range-marching models and used in papers[6–9] with the KZK model.

The appropriate linear wave equation for stratified media is

$$\partial_t^2 \rho' = \nabla^2 p' - \rho_0^{-1} \nabla \rho_0 \cdot \nabla p' \tag{A1}$$

where $\rho_0(h)$ is the ambient density profile, and h is the altitude above the earth. We adopt a moving coordinate system with the primary propagation direction x at an arbitrary angle α (positive upward) to the horizontal (Fig. 1), so that

$$\frac{\partial \rho_0}{\partial x} = \sin\alpha \frac{d\rho_0(h)}{dh}. \tag{A2}$$

The other coordinates (y, z) of the moving coordinate system are chosen so that y is horizontal, and z has a positive vertical component. The vertical extent of the moving coordinate system needs to be sufficient to contain an adequate range of refracted rays from source to ground stations within the dotted lines of Fig. 1.

In Eq. (A1), we substitute

$$\begin{aligned}
\partial_t^2 \rho' &= (D_t - c_0\partial_x)(D_t - c_0\partial_x)\rho' \\
&= \left(D_t^2 - c_0\partial_x D_t - c_0\partial_x(D_t - c_0\partial_x)\right)\rho' \\
&\simeq -2c_0\partial_x D_t\rho' + c_0\partial_x(c_0\partial_x\rho') \\
&= -2c_0\partial_x D_t\rho' + c_0 c_{0x}\rho'_x + c_0^2\rho'_{xx}.
\end{aligned} \tag{A3}$$

In Eq. (A3), we have used the fact that the operator $(D_t - c_0 \partial_x) = \partial_t$ commutes with $c_0\partial_x$, and introduced the notation subscript x for ∂_x. In the second line of Eq. (A3), we have dropped $D_t^2\rho'$ as being smaller than the other terms on the right. Now with $p'(\mathbf{r}) = c_0^2(h)\rho'(\mathbf{r})$, we supress transverse derivatives and express Eq. (A1) to lowest order in environmental gradients as

$$\partial_t^2 \rho' = c_0^2\rho'_{xx} + 2\rho'_x \cdot 2c_0c_{0x} - c_0^2\rho'_x\rho_{0x}/\rho_0. \tag{A4}$$

Combining Eqs. (A3) and (A4), then carrying out an x integral, the reduced 1D wave equation is

$$\begin{aligned}
D_t\rho' &= \int^x \frac{1}{2}\left(\rho'_x c_0 \rho_{0x}/\rho_0 - 3\rho'_x c_{0x}\right)dx \\
&\simeq \frac{1}{2}\rho' c_0(\rho_0 - 3c_{0x}/c_0).
\end{aligned} \tag{A5}$$

The last expression in Eq. (A5) results from integration by parts, dropping second order environmental gradients. Equation (A5) together with $D_t\rho_0 = c_0\rho_{0x}$ and $c_0^2 = \gamma p_0/\rho_0$ can be used to arrive at a one dimensional linear equation for the normalized overdensity (for constant γ). Define

$$\tilde{\rho}' \equiv \frac{\rho'(\mathbf{r}, t)}{\rho_0(h_0)} \tag{A6}$$

where $\rho_0(h_0)$ is the ambient density evaluated at the center of the moving computational window. Then

$$\begin{aligned}
D_t\tilde{\rho}' &= -\frac{1}{2}c_0\frac{\rho'}{\rho_0}\partial_x \ln\left(\rho_0 c_0^3\right) \\
&= -\frac{1}{2}c_0\frac{\rho'}{\rho_0}\partial_x \ln(p_0 c_0).
\end{aligned} \tag{A7}$$

Now we turn to a derivation of Eq. (8). Leonard Euler showed that at an arbitrary point O on a smooth surface, there are two orthogonal principal directions along which the curvature of the surface reaches an extremum. We may model such a surface as

$$u^2 = x^2 + ay^2 + bz^2 = \text{const} \tag{A8}$$

where (y, z) are the principal directions, and point O is taken so that $x > 0$, $y = z = 0$, so that $\hat{x}$ is normal to the surface at O. In Eq. (A8), a and b are dimensionless constants (not necessarily positive). Let us calculate $\nabla^2 f(u)$:

$$\nabla^2 f(u) = \nabla \cdot f_u \nabla u = f_u \nabla^2 u + \nabla f_u \cdot \nabla u.$$

At point O, we have $\nabla u = \hat{x}$ and $\nabla^2 f(u) = f_x \nabla^2 u + f_{xx}$. From Eq. (A8) it follows that at O, $\nabla^2 u = (a+b)/x$. Consider the curve $u = \text{const}$ on $z = 0$: $x^2 + ay^2 = c^2$, with c constant. The radius of curvature at $y = 0$ is $r_1 = x/a$. Similarly, the radius of curvature of $x^2 + bz^2$ at $z = 0$ is $r_2 = x/b$. Thus at point O,

$$\nabla^2 u = (a+b)/x = r_1^{-1} + r_2^{-1} \tag{A9}$$

so that at point O

$$\begin{aligned}
&\nabla^2 f(u) = f_{xx} + \left(r_1^{-1} + r_2^{-1}\right)f_x \quad \text{or} \\
&\nabla_\perp^2 f = \left(r_1^{-1} + r_2^{-1}\right)f_x.
\end{aligned} \tag{A10}$$

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