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On Semigroup Rings

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ON SEMIGROUP RINGS

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This thesis presents some properties of semigroup rings. The main considerations are directed toward determining properties of semigroup rings that can be related to properties of the semigroup or ring involved. The major theorems on semigroup rings are presented in detail. A computer program written in the Symbolic Programming System (SPS) for the IBM 1620 which generates the addition and multiplication tables for semigroup rings is included.

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CHAPTER I

THE PROBLEM AND DEFINITIONS

I. THE PROBLEM

A semigroup ring is a system of functions from a semigroup into a ring which forms a ring under suitable definitions for the operations.

The current trend in this world of increasing specialization is to learn as much as possible about some small area of an important field. It is in this respect that semigroup rings are important. They comprise a small subclass of important algebraic systems called rings. Semigroup rings can be considered as polynomial rings as illustrated by a later example. They can also be considered as a system of functions, a central concept in mathematics.

Some of the properties of semigroup rings are described, illustrated and proved in this study. In particular, the central problem focuses on how various properties of the semigroup and ring are reflected in the resulting semigroup ring.

Before formulating properties of an abstract system, some examples are usually considered. For finite orders, the order of a semigroup ring equals the order of the ring raised to the power of the order of the semigroup involved.

Hence, for even small order semigroups and rings, the resulting semigroup ring will have relatively large order. The difficulty of constructing large complex examples in any reasonable length of time was solved by designing a computer program which generates the addition and multiplication tables for semigroup rings.

II. DEFINITIONS

Before defining a semigroup ring, some basic definitions of elementary abstract systems used in this study are formulated.

Semigroup. A semigroup $(S,\cdot)$ is a system consisting of a non-empty set S and a binary operation $\cdot$ on S such that $\cdot$ is associative, i.e., $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ for all $a,b,c\in S$.

When there is no chance of confusion, $a\cdot b$ will be written ab .

Group. A group $(G,+)$ is a system consisting of a non-empty set G and a binary operation $+$ on G satisfying the following conditions:

(i) $+$ is associative,

(ii) there is an element 0 in G called the right identity, such that $a + 0 = a$ for all $a\in G$, and

(iii) for every element $a\in G$ there exists an element $-a\in G$ called the right inverse of a , such that $a + (-a) = 0$.

Any group in which the operation $+$ is commutative, i.e., $a + b = b + a$ for all $a,b \varepsilon G$, is called an Abelian group.

Ring. A ring $(R,+,\cdot)$ is a system consisting of a non-empty set R and two binary operations $+$ and $\cdot$ on R such that

(i) $(R,+)$ is an Abelian group,
(ii) $(R,\cdot)$ is a semigroup, and
(iii) both the left and right distributive laws hold, i.e., $a\cdot(b+c) = a\cdot b + a\cdot c$ and $(b+c)\cdot a = b\cdot a + b\cdot c$ for all $a,b,c,\varepsilon R$.

In any ring $(R,+,\cdot)$, the Abelian group $(R,+)$ is called the additive structure, and the semigroup $(R,\cdot)$ is called the multiplicative structure.

Commutative ring. A ring $(R,+,\cdot)$ is a commutative ring if $a\cdot b = b\cdot a$ for every $a,b\varepsilon R$.

Ring with identity. A ring $(R,+,\cdot)$ is a ring with identity if there is an element 1 in R such that $a\cdot 1 = 1\cdot a = a$ for all $a\varepsilon R$.

Zero ring. A ring $(R,+,\cdot)$ is a zero ring if $a\cdot b = 0$ for all $a,b\varepsilon R$.

Integral domain. A commutative ring with identity $(R,+,\cdot)$ is an integral domain if $a\cdot b = 0$ implies $a = 0$ or $b = 0$ for all $a,b\varepsilon R$. If this condition holds, the system is said to be free of zero divisors.

Throughout this thesis, S, G and R will be used to represent a semigroup, group and ring, respectively.

Jacobsen's definition (6:95) of a semigroup ring will be used in this thesis.

<u>Semigroup ring</u>. Let $(S,\cdot)$ be any semigroup and $(R,+,\cdot)$ be any ring. Let T be the set of all functions defined on S with values in R such that $f(s) = 0$ for all but a finite number of $s \varepsilon S$. Further, define addition (pointwise) and multiplication (convolution) on T as follows:

$$\text{(i)} \quad (f_1 \oplus f_2)\,(s) = f_1(s) + f_2(s),$$

and

$$\text{(ii)} \quad (f_1 \odot f_2)\,(s) = \sum_{uv=s} f_1(u)\cdot f_2(v)$$

where f_1, $f_2 \varepsilon T$. The symbol $\sum\limits_{uv=s}$ indicates the summation is taken over all pairs u,v of elements whose product is s. The set T equipped with the operations $\oplus$ and $\odot$ is called the semigroup ring of S with respect to R and denoted $(T,\oplus,\odot)$. A typical element $f \varepsilon T$ will be represented by $\{f_\alpha\}_{\alpha\varepsilon S}$ where $f(\alpha) = f_\alpha$.

III. ORGANIZATION OF REMAINDER OF THE THESIS

The first theorem contains a fundamental result about semigroup rings, a semigroup ring is a ring.

In the next section, two examples of semigroup rings are examined in order to better understand their structure.

The main body of the thesis contains several theorems relating properties of a semigroup ring to properties of the underlying semigroup and ring. These theorems culminate with a theorem giving necessary and sufficient conditions for a semigroup ring to be an integral domain.

The computer program used to generate semigroup rings is presented in the last chapter with flow charts and a detailed explanation of the logic involved. A complete printout of the program and an example of its output are included in the appendix.

CHAPTER II

REVIEW OF THE LITERATURE

As indicated earlier, Jacobsen's definition (6:95) of a semigroup ring is used in this thesis. There is another definition by Redei (9:225) which emphasizes the concepts of an algebra and a vector space in formulating the definition of a semigroup ring.

There has been very little work published on semigroup rings. There is a paper by Hans Schneider and Julian Weissglass (10:1) entitled "Group Rings, Semigroup Rings and Their Radicals". As indicated by the title of this paper, semigroup rings and group rings are closely related. Group rings differ from semigroup rings in that the domain of the functions under consideration is a group instead of a semigroup. The rings resulting from this change necessarily have more structure due to the increased structure in the domain. In the field of group rings many papers have been published. Relative to group rings, Lambeck (7:172) has an extensive bibliography as well as some interesting results on group rings in the appendix. There are two papers by D. B. Coleman (2:962; 3:511) that concern group rings.

There is extensive literature on semigroups, especially the work of Clifford and Preston (1:2-224).

McCoy (8:1-46) contains a comprehensive treatment of rings.

There is an abstract by Robert Gilmer (4:21) stating necessary and sufficient conditions for a semigroup ring to be an integral domain. A complete proof of his theorem is contained in this thesis.

CHAPTER III

INTRODUCTION TO SEMIGROUP RINGS

The first theorem begins the study of semigroup rings by establishing that semigroup rings are rings.

Theorem 3-1. The semigroup ring T of S with respect to R is a ring.

Proof. Clearly, pointwise addition and convolution multiplication of two functions from S into R result in a function from S into R.

To show the associativity of $\oplus$, let f_1, f_2, f_3 be elements of T. Then

$$\begin{aligned}
((f_1 \oplus f_2)) \oplus f_3(s) &= (f_1 \oplus f_2)(s) + f_3(s) \\
&= (f_1(s) + f_2(s)) + f_3(s) \\
&= f_1(s) + (f_2(s) + f_3(s)) \\
&= f_1(s) + ((f_2 \oplus f_3)(s)) \\
&= (f_1 \oplus (f_2 \oplus f_3))(s)
\end{aligned}$$

using the definition of $\oplus$ in T and the associativity of $+$ in the ring R. Therefore,

$$(f_1 \oplus f_2) \oplus f_3 = f_1 \oplus (f_2 \oplus f_3) ,$$

which completes the proof of associativity of $\oplus$ in T.

For an additive identity, consider the function $\Theta(s)=0$ for all $s \varepsilon S$, where 0 is the additive identity in R. Then for any function $f \varepsilon T$,

$$
\begin{aligned}
(f \oplus \Theta)(s) &= f(s) + \Theta(s) \\
&= f(s) + 0 \\
&= f(s) .
\end{aligned}
$$

Hence, Θ is the right identity for T since $f \oplus \Theta = f$ for all $f \varepsilon T$.

For the additive inverse of any function $f \varepsilon T$, consider the function $-f$ defined by $(-f)(s) = -(f(s))$ where $[-(f(s))]$ is the additive inverse of $f(s)$ in R.

Then

$$
\begin{aligned}
[f \oplus (-f)](s) &= f(s) + [-(f(s))] \\
&= 0
\end{aligned}
$$

for all $s \varepsilon S$. Therefore, $f \oplus (-f) = \Theta$. This implies that $-f$ is the right inverse of f.

To complete the proof that $(T,\oplus)$ is an Abelian group, let $f_1, f_2 \varepsilon T$. Then

$$
\begin{aligned}
(f_1 \oplus f_2)(s) &= f_1(s) + f_2(s) \\
&= f_2(s) + f_1(s) \\
&= (f_2 \oplus f_1)(s) ,
\end{aligned}
$$

since addition is commutative in R. Therefore,

$f_1 \oplus f_2 = f_2 \oplus f_1$ for all $f \varepsilon T$. Hence $\oplus$ is commutative in T.

To show that $\odot$ is asociative, let $f_1, f_2, f_3 \varepsilon T$. Consider,

$$\begin{aligned}
\big((f_1 \odot f_2)\big)\odot f_3\big)(s) &= \sum_{uv=s} (f_1 \odot f_2)(u)\cdot f_3(v) \\
&= \sum_{uv=s} \Big(\sum_{rt=u} f_1(r)\cdot f_2(t) \Big) \cdot f_3(v) \\
&= \sum_{(rt)v=s} \big(f_1(r)\cdot f_2(t)\big) \cdot f_3(v) \\
&= \sum_{rtv=s} f_1(r)\cdot f_2(t)\cdot f_3(v) \ ,
\end{aligned}$$

since multiplication in both the semigroup and ring is associative. But,

$$\begin{aligned}
\big(f_1 \odot (f_2 \odot f_3)\big)(s) &= \sum_{uv=s} f_1(u)\cdot(f_2 \odot f_3)(v) \\
&= \sum_{uv=s} f_1(u)\cdot\Big(\sum_{xy=v} f_2(x)\cdot f_3(y)\Big) \\
&= \sum_{u(xy)=s} f_1(u)\cdot\big(f_2(x)\cdot f_3(y)\big) \\
&= \sum_{uxy=s} f_1(u)\cdot f_2(x)\cdot f_3(y) \ .
\end{aligned}$$

Since both summations are over all triple factorizations of s ,

$$\sum_{rtv=s} f_1(r)\cdot f_2(t)\cdot f_3(v) = \sum_{uxy=s} f_1(u)\cdot f_2(x)\cdot f_3(y) \ ,$$

which implies that $(f_1 \odot f_2) \odot f_3 = f_1 \odot (f_2 \odot f_3)$ for all $f_1, f_2, f_3 \varepsilon T$. Therefore, $(T,\odot)$ is a semigroup.

To complete the proof, the distributive law must be established. Let f_1, f_2, $f_3 \in T$. Then

$$\begin{aligned}
\big(f_1 \odot (f_2 \oplus f_3)\big)(s) &= \sum_{uv=s} f_1(u)\cdot(f_2 \oplus f_3)(v) \\
&= \sum_{uv=s} f_1(u)\cdot(f_2(v) + f_3(v)) \\
&= \sum_{uv=s} f_1(u)\cdot f_2(v) + f_1(u)\cdot f_3(v) \\
&= \sum_{uv=s} f_1(u)\cdot f_2(v) + \sum_{uv=s} f_1(u)\cdot f_3(v) \\
&= (f_1 \odot f_2)(s) + (f_1 \odot f_3)(s) \\
&= \big((f_1 \odot f_2) \oplus (f_1 \odot f_3)\big)(s) ,
\end{aligned}$$

using properties of the ring R . This implies that $f_1 \odot (f_2 \oplus f_3) = (f_1 \odot f_2) \oplus (f_1 \odot f_3)$ which completes the proof that the left distributive property holds in T.

The right distributive property is proved similarly.

Thus $(T,\oplus)$ is an Abelian group, $(T,\odot)$ is a semigroup and both the left and right distributive laws hold. Therefore, by the definition of a ring, $(T, \oplus, \odot)$ is a ring.

It should be noted that the finitely non-zero restriction on the functions in T is necessary to insure that the summation in the definition of multiplication be finite. This could be replaced with the restriction on the semigroup that every element have only a finite number of distinct factors. This, however, would restrict the class of semigroups used in the definition of semigroup ring since there are semigroups whose elements have an infinite number of factors.

A good way to study an abstract system is to construct some examples. Example 3-1 illustrates the connection between semigroup rings and polynomial rings.

Before examining this connection, consider the following definition of a polynomial ring.

Polynomial ring. Let $(R,+,\cdot)$ be a ring and $(I,+)$ be the non-negative integers. A polynomial p in an indeterminate x over the ring R is a formal sum $\sum_{i\varepsilon I} a_i x^i$, where $a_i \varepsilon R$ and only a finite number of the a_i's are different from zero. Two polynomials, $p = \sum_{i\varepsilon I} a_i x^i$ and $q = \sum_{i\varepsilon I} b_i x^i$ are equal if and only if $a_r = b_r$ for all $r \varepsilon I$.

Define addition (pointwise) and multiplication (convolution) on $R[x]$, the set of all polynomials over R, by

$$\text{(i)} \quad p + q = \sum_{i\varepsilon I} (a_i + b_i) x^i ,$$

and

$$\text{(ii)} \quad p \cdot q = \sum_{i\varepsilon I} c_i x^i$$

where $c_r = \sum_{i+j=r} a_i b_j$. With this definition, $(R[x],+,\circ)$ is a ring (6:92).

Example 3-1. Let $(S,+)$ be the semigroup of non-negative integers and $(R,+,\cdot)$ be any ring. The resulting semigroup ring $(T,\oplus,\odot)$ is isomorphic to the ring of polynomials $R[x]$ in an indeterminant x with coefficients in R and exponents in S.

Consider the mapping Φ between the semigroup ring T and the polynomial ring $R[x]$ defined as follows:

$$\Phi(f) = \sum_{s\varepsilon S} f_s x^s ,$$

where $f = \{f_s\}_{s\varepsilon S}$.

The mapping Φ is onto, since for any polynomial, $p = \sum_{i\varepsilon S} a_i x^i$ in $R[x]$, $f = \{a_i\}_{i\varepsilon S}$ is in T.

Let f and g be two distinct elements of T such that

$$\Phi(f) = \sum_{i\varepsilon S} a_i x^i = \Phi(g)$$

Then $f_i = a_i = g_i$ for all $i\varepsilon S$. But, $f_i = g_i$ implies that $f(i) = g(i)$ for all $i\varepsilon S$. Hence, $f = g$. Therefore, Φ is one-to-one.

Consider

$$\begin{aligned}\Phi(f \oplus g) &= \sum_{i\varepsilon S} (f \oplus g)_i \, x^i \\ &= \sum_{i\varepsilon S} (f_i \oplus g_i) x^i \\ &= \sum_{i\varepsilon S} f_i x^i + \sum_{i\varepsilon S} g_i x^i\end{aligned}$$

$$= \Phi(f) + \Phi(g) ,$$

using the definition of Φ and the definition of addition in T . Also,

$$\Phi(f \odot g) = \sum_{i \varepsilon S} (f \odot g)_i x^i$$

$$= \sum_{i \varepsilon S} \left(\sum_{s+t=i} (f_s \cdot g_t) x^i \right)$$

$$= \Phi(f) \cdot (g) ,$$

by definition of multiplication in $R[x]$.

Now Φ is a one-to-one operation preserving mapping from T onto $R[x]$.

In general, any semigroup ring can be thought of in terms of its isomorphic polynomial ring under the above identification. This is of little value however, unless the semigroup is totally ordered so that the concept of the degree of a polynomial can be introduced.

The next example is finite and can be illustrated with Cayley tables.

<u>Example 3-2</u>. Let the semigroup S be the zero semigroup of order two. Its Cayley table is

$\cdot$	s	t
s	s	s
t	s	s

.

Let the ring R be the system of integers modulo 2 . Its Cayley tables are:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

.

The four functions in the resulting semigroup ring are f_1, f_2, f_3, f_4 as defined below for $s,t \in S$.

$$f_1(x) = 0 \text{ for all } x \in S ,$$

$$f_2(x) = \begin{cases} 0 \text{ if } x=S, \\ 1 \text{ if } x=t, \end{cases}$$

$$f_3(x) = \begin{cases} 1 \text{ if } x=s, \\ 0 \text{ if } x=t, \end{cases} \text{ and}$$

$$f_4(x) = 1 \text{ for all } x \in S.$$

The additive structure of the resulting semigroup ring T is Klein's four group with Cayley table

$\oplus$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_3	f_4	f_1
f_3	f_2	f_4	f_1	f_2
f_4	f_4	f_1	f_2	f_3

.

The multiplicative structure of T is given by the Cayley table:

$\odot$	f_1	f_2	f_3	f_4
f_1	f_1	f_1	f_1	f_1
f_2	f_1	f_3	f_3	f_1
f_3	f_1	f_3	f_3	f_1
f_4	f_1	f_1	f_1	f_1

.

CHAPTER IV

THEOREMS ON SEMIGROUP RINGS

I. THE ADDITIVE STRUCTURE

Let $(R,+)$ be the additive structure of a ring. Consider the formal products $\prod_{i\varepsilon I} R_i$ for any index set I and $R_i = R$. The sum of two elements in this system is defined pointwise by

$$\{a_i\}_{i\varepsilon I} + \{b_i\}_{i\varepsilon I} = \{c_i\}_{i\varepsilon I}$$

where $c_i = a_i + b_i$. This system is called the Cartesian product group. If there are only finitely many non-zero coordinates in each element of the system, i.e., for $\{a_i\}_{i\varepsilon I} \ \varepsilon \prod_{i\varepsilon I} R_i$, $a_i = 0$ for all but a finite number of a_i's, then the system is called the direct product group D.

In the direct product group, the additive identity is $\{a_i\}_{i\varepsilon I}$, where $a_i = 0$ for all $i\varepsilon I$.

The additive inverse of an element $\{a_i\}_{i\varepsilon I}$ in D is $\{-a_i\}_{i\varepsilon I}$, where $-a_i$ is the additive inverse of a_i in R.

Addition in D is commutative and associative since addition in R is commutative and associative. Hence, D is an Abelian group.

The next theorem relates the direct product group to this study of semigroup rings.

Theorem 4-1. The additive structure of the semigroup ring T of S with respect to R is isomorphic to the direct product group of (R,+) with S as the index set.

Proof. Let Φ be a mapping from T onto D such that $\Phi(f) = f$ where $f = \{f_\alpha\}_{\alpha \in S}$ and $f_\alpha = f(\alpha)$. Then Φ is an isomorphism from T onto D.

This completely determines the additive structure of the semigroup ring. When the semigroup is finite, say of order n, the semigroup ring's additive structure is exactly n copies of the additive structure of the ring.

For further results on direct and Cartesian product groups, see Hall (5:33).

II. THE MULTIPLICATIVE STRUCTURE

Because of the complex multiplication involved in a semigroup ring, and the fact that the multiplication need only be associative, the multiplicative structure of a semigroup ring is not easily characterized. This makes the multiplicative structure far more interesting.

This section is composed of several theorems relating various properties of the semigroup and the ring to properties of the semigroup ring. The first theorems are characterization theorems because they deal with properties of the semigroup and ring which do not yield a system with more structure than a ring. The last theorems are generalization theorems since they are concerned with properties of the semigroup and ring which yield more structure than a ring for the resulting semigroup ring. Together they culminate with Gilmer's theorem on integral domains (4:21).

The first theorem characterizes some of the annihilators in a semigroup ring. A left [right] annihilator is an element f where $f \odot g = 0$ [$g \odot f = 0$] for all g in T. Unfortunately, the theorem does not necessarily describe all of the annihilators in a semigroup ring.

Theorem 4-2. Any function in T whose range is composed entirely of left [right] annihilators in the ring R is a left [right] annihilator of the semigroup ring T of S with respect to R.

Proof. Let $f = \{f_\alpha\}_{\alpha \varepsilon S}$ be an element of T where the f_α's are left annihilators in the ring R. Then for any $g \varepsilon T$,

$$(f \circledcirc g)(s) = \sum_{uv=s} f(u)\cdot g(v)$$
$$= \sum_{uv=s} f_u \cdot g(v)$$
$$= 0\ ,$$

since f_u is a left annihilator in R for every $u \varepsilon S$. Hence, f is a left annihilator in T. The case in which each of the f_α's are right annihilators yields a right annihilator in T in a similar way.

Notice that the converse of this theorem is not true. This is shown by example 3-2 where $f_i \circledcirc f_4 = f_4 \circledcirc f_i = 0$ for all f_i in the semigroup ring, but the range of f_4 is $\{1\}$, which is not an annihilator in the ring of integers modulo 2.

Theorem 4-3 gives necessary and sufficient conditions for a semigroup ring to be a zero ring, i.e., a ring in which $a \cdot b=0$ for all a,b in the ring.

<u>Theorem 4-3</u>. The semigroup ring T of S with respect to R is a zero ring if and only if the ring R is a zero ring.

<u>Proof</u>. Assume the semigroup ring T is a zero ring. Then $f \circledcirc g = 0$ for all $f,g \varepsilon T$.

Consider the two functions $f,g \varepsilon T$ defined by

$$f(s) = \begin{cases} 0 \text{ if } & s \neq u_o , \\ a \text{ if } & s = u_o , \end{cases}$$

and

$$g(s) = \begin{cases} 0 \text{ if } & s \neq v_o , \\ b \text{ if } & s = v_o , \end{cases}$$

where a and b are any two non-zero elements of R. If there are not two non-zero elements in R, then we are done since R would then be a zero ring. Further denote the product $u_o v_o$ in S by s_o. Now $(f \odot g)(s_o) = 0$ since T is a zero ring. But,

$$\begin{aligned} (f \odot g)(s_o) &= \sum_{uv=s_o} f(u) \cdot g(v) \\ &= 0 + f(u_o) \cdot g(v_o) \\ &= a \cdot b \quad . \end{aligned}$$

Therefore, $a \cdot b = 0$ for all $a, b \varepsilon R$. Hence R is the zero ring.

Now assume that R is a zero ring. Then $a \cdot b = 0$ for all $a, b \varepsilon R$. Let f and g be any two functions in T. Then

$$\begin{aligned} (f \odot g)(s_o) &= \sum_{uv=s_o} f(u) \cdot g(v) \\ &= 0 \end{aligned}$$

since $f(u) \cdot g(v) = 0$ for all $u, v \varepsilon S$ because $f(u), g(v) \varepsilon R$. Therefore, T is a zero ring.

The next theorem gives necessary and sufficient conditions for a semigroup ring to be a commutative ring.

Theorem 4-4. Let R be a ring which is not a zero ring. The semigroup ring T of S with respect to R is a commutative ring if and only if the semigroup S and the ring R are commutative.

Proof. Assume that the ring R is not commutative. Consider the functions f and g in T defined for some fixed $u_o \varepsilon S$ as

$$f(s) = \begin{cases} 0 \text{ if } & s \neq u_o , \\ a \text{ if } & s = u_o , \end{cases}$$

and

$$g(s) = \begin{cases} 0 \text{ if } & s \neq u_o , \\ b \text{ if } & s = u_o , \end{cases}$$

where a and b are two non-zero elements in R such that $ab \neq ba$. Such a pair exists since R is not commutative. Denote the product $u_o u_o$ by $s_o \varepsilon S$. Then

$$\begin{aligned}(f \oplus g)(s_o) &= \sum_{uv=s_o} f(u)\cdot g(v) \\ &= f(u_o)\cdot g(u_o) + \sum_{\substack{uv=s_o \\ u,v\neq u_o}} f(u)\cdot g(v) \\ &= ab + 0 \\ &= ab,\end{aligned}$$

and

$$(g \odot f)(s_o) = \sum_{uv=s_o} g(u)\cdot f(u_o)$$

$$= 0 + g(u_o)\cdot f(u_o)$$

$$= ba \quad .$$

But since $ab \neq ba$, $(f \odot g)(s_o) \neq (g \odot f)(s_o)$. Therefore, $f \odot g \neq g \odot f$ and T is not commutative.

Now assume the semigroup is not commutative. Then $u_o v_o \neq v_o u_o$ for some $u_o, v_o \varepsilon S$. Clearly $u_o \neq v_o$. Then examine functions

$$f(s) = \begin{cases} 0 \text{ if } & s \neq u_o \ , \\ a \text{ if } & s = u_o \ , \end{cases}$$

and

$$g(s) = \begin{cases} 0 \text{ if } & s \neq v_o \ , \\ b \text{ if } & s = v_o \ , \end{cases}$$

where $a, b \varepsilon R$ such that $a\cdot b \neq 0$. Such a pair exists since R is not the zero ring. Let s_o denote the product $u_o v_o$ in S .

Consider

$$(f \odot g)(s_o) = \sum_{uv-s_o} f(u)\cdot g(v)$$

$$= f(u_o)\cdot g(v_o) + \sum_{\substack{uv=s_o \\ u\neq u_o \\ v\neq v_o}} f(u)\cdot g(v)$$

$$= ab + 0$$

$$= ab \quad .$$

But,

$$(g \Theta f)(s) = \sum_{uv=s_o} g(u)\cdot f(v)$$

$$= \sum_{\substack{uv=s_o \\ u\neq v_o \\ v\neq u_o}} g(u)\cdot f(v)$$

since $v_o\cdot u_o\neq s_o$. Therefore, $(g \Theta f)(s_o) = 0$. Hence, $(f \Theta g)(s_o) \neq (g \Theta f)(s_o)$, which implies that T is not commutative. This completes the first part of the proof using the method of contraposition.

For the proof of the sufficient conditions, assume both S and R are commutative. Then for any functions $f,g\varepsilon T$,

$$(f \Theta g)(s) = \sum_{uv=s} f(u)\cdot g(v)$$

$$= \sum_{uv=s} g(v)\cdot f(u)$$

$$= \sum_{vu=s} g(v)\cdot f(u)$$

$$= (g \Theta f)(s) .$$

Therefore, $f \Theta g = g \Theta f$. Hence T is a commutative ring.

One should note that the commutativity of the ring or the semigroup alone is not sufficient to insure the commutativity of the semigroup ring. The following example is an illustration of this fact.

Example 4-1. Let the semigroup be defined by the following Cayley table:

$\cdot$	s	t
s	s	t
t	s	t

,

and let the ring be the system of integers modulo 2. The semigroup is not commutative. The multiplicative structure of the semigroup ring is given by the Cayley table

$\odot$	f_1	f_2	f_3	f_4
f_1	f_1	f_1	f_1	f_1
f_2	f_1	f_2	f_3	f_4
f_3	f_1	f_2	f_3	f_4
f_4	f_1	f_1	f_1	f_1

.

The semigroup ring is clearly not a commutative ring.

For an example of a semigroup ring that is not a commutative ring because the ring is not commutative, see the example in the appendix.

Theorem 4-5. If R is a ring with identity and S is a semigroup with identity then the semigroup ring T of S with respect to R is a ring with identity.

Proof. Let e_r be the identity element for R and and e_s be the identity for S. Consider the following function

$$\iota(s) = \begin{cases} e_r & \text{if } s = e_s , \\ 0 & \text{if } s \neq e_s . \end{cases}$$

Then for any function $f \in T$,

$$\begin{aligned}(f \odot \iota)(s) &= \sum_{uv=s} f(u) \cdot \iota(v) \\ &= f(s) \cdot (e_s) + \sum_{\substack{uv=s \\ v \neq e_s}} f(u) \cdot \iota(v) \\ &= f(s) \cdot e_r + 0 \\ &= f(s) \quad .\end{aligned}$$

Similarly $(\iota \odot f)(s) = f(s)$. Therefore, $f \odot \iota = \iota \odot f = f$ for all $f \in T$. This completes the proof that T is a ring with identity.

Corollary 4-1. If R is a ring with a left [right] identity and S is a semigroup with a left [right] identity, then the semigroup ring T of S with respect to R has a left [right] identity.

Proof. Consider the function $e \in T$ defined by

$$e(s) = \begin{cases} e_r & \text{if } s = e_s \\ 0 & \text{if } s \neq e_s \end{cases} ,$$

where e_s is the left [right] identity in the semigroup and e_r is the left [right] identity in the ring.

Then e is a left [right] identity for T.

Theorem 4-6. Let S be a cancellative semigroup. If the semigroup ring T of S with respect to R is a ring with identity, then the ring R and the semigroup S have identities.

Proof. The proof of this theorem consists of the next three lemmas.

Lemma 4-6 (a) If R is a ring without identity and S is a semigroup with identity, then the semigroup ring T of S with respect to R is a ring without identity.

Proof. Assume R is a ring without identity and S has an identity element 1 such that $1 \cdot s = s \cdot 1 = s$ for all $s \in S$. We want to show that T does not have an identity. We proceed indirectly assuming T has an identity ι such that $\iota \odot f = f \odot \iota = f$ for all $f \in T$ and derive a contradiction. Consider the function $f \in T$ defined as

$$f(s) = \begin{cases} a & \text{if } s = s_o , \\ 0 & \text{if } s \neq s_o , \end{cases}$$

where s_o is a fixed element of S. Then

$$
\begin{aligned}
(\iota \odot f)(s_0) &= \sum_{uv=s_0} \iota(u)\cdot f(v) \\
&= \iota(1)\cdot f(s_0) + \sum_{\substack{uv=s_0 \\ v\neq s_0}} \iota(u)\cdot f(v) \\
&= \iota(1)\cdot a,
\end{aligned}
$$

for each $s \in R$. But

$$
\begin{aligned}
(\iota \odot f)(s_0) &= f(s_0) \\
&= a .
\end{aligned}
$$

Therefore, $\iota(1)\cdot a = a$ for all $a \in R$. A similar argument for $f \odot \iota$ yields $a\cdot \iota(1) = a$ for all $a \in R$. Thus $\iota(1)$ is an identity in R. This contradicts the assumption that R was a ring without identity completing the proof. (Notice that the assumption that S is cancellative was not needed in this lemma.)

<u>Lemma 4-6 (b)</u>. If R is a ring with identity and S is a concellative semigroup without identity, then the semigroup ring T of S with respect to R is a ring without identity.

<u>Proof</u>. Assume that R is a ring with identity 1 such that $1\cdot a = a\cdot 1 = a$ for all $a \in R$, and that S is a ring without identity. In order to prove that T is a ring without identity we will proceed indirectly by assuming T has an identity ι such that $\iota \odot f = f \odot \iota = f$ for all $f \in T$ and derive a contradiction. These assumptions require

that for all $s \varepsilon S$ there must exist a $u \varepsilon S$ such that $u \cdot s = s$ and $\iota(u) = 1$. The requirement that $\iota(u) = 1$ for any two distinct $u \varepsilon S$ will yield a contradiction of the single-valuedness of elements of T.

Given an $s_o \varepsilon S$, consider the function $f \varepsilon T$ defined by

$$f(s) = \begin{cases} \theta & \text{if } s \neq s_o \ , \\ 1 & \text{if } s = s_o \ . \end{cases}$$

Then

$$\begin{aligned} (\iota \odot f)(s_o) &= (f \odot \iota)(s_o) \\ &= f(s_o) \\ &= 1 \ . \end{aligned}$$

By the definition of multiplication in T ,

$$(\iota \odot f)(s_o) = \sum_{uv=s_o} \iota(u) \cdot f(s) \ .$$

Hence, there exists a $u_o \varepsilon S$ such that $u_o s_o = s_o$. This u_o will be unique since S is cancellative. Then

$$\begin{aligned} (\iota \odot f)(s_o) &= \iota(u_o) \cdot f(s_o) + \sum_{\substack{uv=s_o \\ v \neq s_o}} \iota(u) \cdot f(v) \\ &= \iota(u_o) \cdot 1 + 0 \\ &= \iota(u_o) \ , \end{aligned}$$

in which case $\iota(u_o) = 1$. This must be true for every element in S. That is, for every $s_o \varepsilon S$ there exists an element $u_o \varepsilon S$ (it may depend on s_o) such that $u_o s_o = s_o$ and $\iota(u_o) = 1$.

By considering $(f \Theta \iota)(s_o)$, we find that for every $s_o \varepsilon S$ there exists a $v_o \varepsilon S$ such that $s_o v_o = s_o$ and $\iota(v_o) = 1$.

Thus for a given $s_o \varepsilon S$ there exist elements $u_o, v_o \varepsilon S$ such that $u_o s_o = s_o v_o = s_o$ and $\iota(u_o) = \iota(v_o) = 1$. Denote the product $u_o v_o$ by t_o. Then, by the same argument, there exists elements $u_1, v_1 \varepsilon S$ such that $u_1 \cdot t_o = t_o \cdot v_1 = t_o$ and $\iota(u_1) = \iota(v_1) = 1$.

Consider the function $g \varepsilon T$ defined by

$$g(s) = \begin{cases} 1 & \text{if } s = v_o \text{ or } s = t_o \\ 0 & \text{otherwise} \end{cases} .$$

Then, since S is cancellative,

$$\begin{aligned} (\iota \Theta g)(t_o) &= \sum_{uv=t_o} \iota(u) \cdot g(v) \\ &= \iota(u_1) \cdot g(t_o) + \iota(u_o) \cdot g(v_o) \\ &\quad + \sum_{\substack{uv=s \\ v \neq t_o \\ v \neq v_o}} \iota(u) \cdot g(v) \end{aligned}$$

$$= 1\cdot 1 + 1\cdot 1 + 0$$
$$= 1 + 1.$$

But $(\iota \odot g)(t_o) = g(t_o) = 1$. Since $1 + 1 \neq 1$ for any ring, this contradicts the single-valuedness of $\iota \odot g$ as a function in T. This completes the proof of the lemma.

Lemma 4-6 (c). If R is a ring without identity and S is a cancellative semigroup without identity, then the semigroup ring T of S with respect to R is a ring without identity.

Proof. Assume that neither S no R has an identity. Then to prove that T does not have an identity, assume that it does and derive a contradiction.

Let ι be the identity in T and $f \in T$ defined by

$$f(s) = \begin{cases} 0 & \text{if } s \neq s_o , \\ a & \text{if } s = s_o , \end{cases}$$

for some fixed $s_o \in S$ and a non-zero element $a \in R$. If there is no non-zero element in R, we are done since T is then a zero ring, hence contains no identity element.

Then $(\iota \odot f)(s_o) = (f \odot \iota)(s_o) = f(s_o) = a$. If $u\cdot s_o \neq s_o$ for any $u \in S$ then

$$(\iota \odot f)(s_o) = \sum_{uv=s_o} \iota(u)\cdot f(v)$$
$$= \sum_{\substack{uv=s_o \\ v\neq s_o}} \iota(u)\cdot f(v)$$
$$= 0 ,$$

which is a contradiction.

If for each $s_o \varepsilon S$ there is an element $u_o \varepsilon S$ (which may depend on s_o) such that $u_o s_o = s_o$, then

$$(\iota \Theta f)(s_o) = \sum_{uv=s_o} \iota(u)\cdot f(v)$$

$$= \iota(u_o)\cdot f(s_o) + \sum_{\substack{uv=s_o \\ v\neq s_o}} \iota(u)\cdot f(v),$$

since S is cancellative. Then

$$(\iota \Theta f)(s_o) = \iota(u_o)\cdot f(s_o) + 0$$

$$= \iota(u_o)\cdot a.$$

This implies that $\iota(u_o)\cdot a = a$ for every $a \varepsilon R$. Since $(f \Theta \iota)(s_o) = a$ yields similarly that $a\cdot \iota(u_o) = a$. This is a contradiction of the assumption that R is a ring without identity because $\iota(u_o)$ is an identity for R. This completes the proof of this lemma.

The proof of lemmas 4-6(a), (b) and (c) complete the proof of theorem 4-6.

Corollary 4-2. Let S be a cancellative semigroup. If the semigroup ring T of S with respect to R has a left [right] neutral then both the semigroup S and the ring R have a left [right] neutral.

Proof. The proof of this corollary is similar to the proof of theorem 4-6 replacing the word identity with left [right] identity.

Theorem 4-6 employs several of the previous theorems in its proof. As the culmination of the study, it gives necessary and sufficient conditions for a semigroup ring to be an integral domain.

<u>Theorem 4-7</u>. Let R be a commutative ring with identity and S be an additive Abelian semigroup with zero. Then necessary and sufficient conditions for the semigroup ring T of S with respect to R to be an integral domain are the following:

(1) R is an integral domain,
(2) S is cancellative, and
(3) if s and t are distinct elements of S and if n is a natural number, then $ns \neq nt$.

Before proceeding with the proof of this theorem, the next theorems establish some of the interesting consequences of these restrictions on S and R. The first is that the semigroup must be infinite or contain only the zero element.

<u>Theorem 4-8</u>. Let $(S,+)$ be a cancellative Abelian semigroup with at least two distinct elements, s and t, such that $ns \neq nt$ for all natural numbers n. Then S is infinite.

Proof. Assume $S = \{a_1, a_2, a_3, \ldots, a_p\}$ is finite. Since S is cancellative, for a given $a \varepsilon S$, the elements $a_i + a_j = a_j + a_i$ are distinct for each $j = 1, 2, \ldots, p$. Then, since S is cancellative, $a_i + x = a_j$ has a unique solution for each $a_i, a_j \varepsilon S$.

In particular, $a_i + x = a_i$ has a unique solution 0_i for each $a_i \varepsilon S$. This 0_i is unique for the system since, if 0_i is the solution of $a_i + x = a_i$ and $0j$ is the solution of $a_j + x = a_j$, then

$$(a_i+a_j) + 0_j = a_i + a_j$$

and

$$(a_i+a_j) + 0i = a_i + a_j ,$$

since S is associative and commutative. Hence, $0_i = 0_j$ since S is cancellative. Thus, there is a unique $0 \varepsilon S$ such that $a_i + 0 = a_i$ for all $a_i \varepsilon S$.

Further, the equation $a_i + x = 0$ has a unique solution for each $a_i \varepsilon S$, i.e., every $a_i \varepsilon S$ has a unique inverse denoted by $-a_i$.

Since S is finite, for each $a_i \varepsilon S$ there is a natural number q such that $qa_i = 0$. For if not, there is an a_j such that $na_j \neq 0$ for every n. Hence, there are distinct natural numbers 1 and m (we can assume $m < 1$) such that $1a_i = ma_i$. Therefore,

$$\underbrace{a_i + a_i + \ldots + a_i}_{l\text{ - terms}} = \underbrace{a_i + a_i + \ldots + a_i}_{m\text{ - terms}} .$$

This implies that

$$\underbrace{a_i + a_i + \ldots + a_i}_{l\text{ - terms}} + \underbrace{(-a_i - a_i - a_i - \ldots - a_i)}_{m\text{ - terms}} = 0$$

Then, using the commutativity and associativity of S, this becomes:

$$\underbrace{a_i + a_i + \ldots + a_i}_{l\text{ - }m\text{ terms}} = 0 .$$

This implies

$$(l-m)a_i = 0$$

for some natural number $l-m \neq 0$. Hence, for each $a_i \varepsilon S$, there is a natural number q such that $qa_i = 0$.

Let $a_i \varepsilon S$ such that $a_i \neq n0$ for every natural number n. But $n0=0$. Hence, $na_i = n0$ which is a contradiction.

Therefore, S is infinite.

Theorem 4-9. If $(S,+)$ is a cancellative Abelian semigroup with a zero element, then S can be embedded in a group G.

Proof. Consider the set $S \times S$. Define a relation on $S \times S$ by

$$(a,b) \sim (c,d) \text{ iff } a + d = b + c .$$

The relation $\sim$ is reflexive since $(a,b) \sim (a,b)$ in view of the commutativity of S.

The relation is symmetric since $(a,b) \sim (c,d)$ if and only if $a + d = b + c$. This implies $c + b = d + a$ since $=$ is symmetric and S is commutative. Hence, $(c,d) \sim (a,b)$.

The relation $\sim$ is transitive since $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ if and only if $a + d = b + c$ and $c + f = d + e$. Then

$$(a+d) + (c+f) = (b+c) + (d+e)$$

and

$$(a+f) + (d+c) = (b+e) + (d+c)$$

by commutativity and associativity in S. Hence, $a + f = b + e$. by the cancellative property of S. This implies $(a,b) \sim (e,f)$.

Therefore, the relation $\sim$ is an equivalence relation on $S \times S$.

Denote the equivalence class of (a,b) by $(\overline{a,b})$ and let

$$G = \{ (\overline{a,b}) : (a,b) \varepsilon S \times S \}$$

Define addition in G by

$$(\overline{a,b}) * (\overline{c,d}) = (\overline{a+c,b+d})$$

Addition is well defined since if $a,b,c,d,e,f,g,h \varepsilon S$ such that $(a,b) \sim (e,f)$ and $(c,d) \sim (g,h)$, then $a+f = b+c$ and $c+b = d+g$. Therefore, $(a+f) + (c+b) = (b+e) + (d+g)$ and

$(a+c) + (f+h) = (b+d) + (e+g)$, since S is commutative and associative. Therefore,

$$(a+c, b+d) \sim (e+g, f+h)$$

which implies

$$\overline{(a+c, b+d)} = \overline{(e+g, f+h)} .$$

Hence,

$$\overline{(a,b)} * \overline{(c,d)} = \overline{(e,f)} * \overline{(g,h)}$$

by definition of $*$.

The identity of G is $\overline{(0,0)} = \{(a,a) : a \varepsilon S\}$, where 0 is the identity in S . Then

$$\begin{aligned} \overline{(a,b)} * \overline{(0,0)} &= \overline{(a+0, b+0)} \\ &= \overline{(a,b)} . \end{aligned}$$

Every element $\overline{(a,b)}$ in G has an inverse $\overline{(b,a)}$ in G since

$$\begin{aligned} \overline{(a,b)} * \overline{(b,a)} &= \overline{(a+b, b+a)} \\ &= \overline{(0,0)} \end{aligned}$$

since $a+b = b+a$.

G is also commutative and associative since S has these properties.

Therefore, G is an Abelian group

Let $\overline{S} = \{\overline{(a,0)} : a \varepsilon S\}$. Then $(\overline{S}, *)$ is a subgroup of G.

Let ϕ be a mapping from S to $\overline{S}$, where

$$\Phi(a) = \overline{(a,0)} \quad .$$

Then

$$\Phi\,(a+b) = \overline{(a+b,0)}$$

$$= \overline{(a,0)} * \overline{(b,0)}$$

$$= \Phi(a) * \Phi(b) \quad .$$

The mapping Φ is onto and one-to-one since, for $\overline{(a,0)}$ in $\overline{S}$, a is in S. Therefore, Φ is an isomorphism of $(S,+)$ onto $(\overline{S},*)$.

Now we have $(S,+)$ embedded in the difference group $(G,*)$ with $\overline{S}$ as the isomorphic copy of S. This completes the proof of the theorem.

The following theorem reveals that G, as defined in theorem 4-9, is torsion free, i.e., none of its non-zero elements has finite order.

<u>Theorem 4-10</u>. Let G be as defined in theorem 4-9. If distinct $s,t \varepsilon S$ implies $ns \neq nt$ for any natural number n, then G is torsion free.

<u>Proof</u>. Let $(\overline{a,b})$ be any non-zero element of G, and assume the order of $(\overline{a,b})$ is some finite number n. Then

$$n(\overline{a,b}) = (\overline{0,0}) \;,$$

where

$$\begin{aligned} n(\overline{a,b}) &= \underbrace{(\overline{a,b}) * (\overline{a,b}) * \ldots * (\overline{a,b})}_{n\text{ - terms}} \\ &= (\overline{\underbrace{a+a+\ldots+a}_{n\text{-terms}}, \underbrace{b+b+\ldots+b}_{n\text{-terms}}}) \\ &= (\overline{na,nb}) . \end{aligned}$$

Hence,

$$(\overline{na,nb}) = (\overline{0,0}) ,$$

which implies that $(\overline{na,nb}) \sim (0,0)$. Therefore, $na+0 = nb+0$ by definition of $\sim$. Hence, $na=nb$. This is a contradiction of the hypothesis since $a \neq b$. Therefore, every element of G has infinite order which means G is torsion free. Since G (as defined in Theorem 4-9) has been shown to be torsion free under the conditions of Theorem 4-9, it can be totally ordered. Before proceeding with this development, the following terms are defined in order to clarify the proof.

Partial order. A set G is said to be partially ordered by a relation ρ if the relation ρ is defined on G and satisfies,

(i) $a\rho a$ for all $a \varepsilon G$ (ρ is reflexive),
(ii) if $a\rho b$ and $b\rho a$ then $a=b$ (ρ is antisymmetric), and
(iii) if $a\rho b$ and $b\rho c$ then $a\rho c$ (ρ is transitive).

Total order. A set G is said to be totally ordered by ρ if G is partially ordered by ρ and for $a,b \varepsilon G$, either $a\rho b$ or $b\rho a$.

A totally ordered set is sometimes called a chain.

Thereom 4-11. Let G be defined as in theorem 4-9. If G is torsion free, then G can be totally ordered.

Proof. Since G is torsion free, for each $\overline{(a,b)} \in G$ $a,b \neq 0$, $n\overline{(a,b)} \neq \overline{(0,0)}$ for any natural number n. Therefore,

$$N = \{ \overline{(a,b)}, 2\overline{(a,b)}, 3\overline{(a,b)}, \ldots \}$$

is a subsemigroup of G.

Consider all subsemigroups of G which contain N but not $\overline{(0,0)}$. These can be partially ordered by set inclusion. For any chain of such subsemigroups $\{ N_i \}$, $\bigcup_i N_i$ is a subsemigroup of G containing N but not $\overline{(0,0)}$.

Therefore, each chain has an upper bound. Zorn's lemma now assures the existence of a maximal element for the class of all subsemigroups containing N but not $\overline{(0,0)}$. This maximal element P is a subsemigroup containing N but not $\overline{(0,0)}$.

The claim is that for any $\overline{(c,d)} \in G$, exactly one of $\overline{(c,d)} = \overline{(0,0)}$ or $\overline{(c,d)} \in P$ or $\overline{(a,c)} \in P$ holds. If $\overline{(c,d)} = \overline{(0,0)}$, then $\overline{(c,d)} = \overline{(d,c)} \notin P$. If $\overline{(c,d)} \neq \overline{(0,0)}$, then $c \neq d$. If both $\overline{(c,d)}$, and $\overline{(d,c)}$ were elements of P, then their sum $\overline{(c,d)} * \overline{(d,c)} = \overline{(0,0)}$ would have to be in P since P is a subsemigroup of G, hence closed. This is a contradiction since $(\overline{0,0}) \notin P$ by construction of P. Therefore, at most one of $(\overline{c,d})$ and $(\overline{d,c})$ may be in P.

Suppose neither $(\overline{c,d})$ or $(\overline{d,c})$ were in P. Then consider the subsemigroup P_{cd} generated by $(\overline{c,d})$ and P and the subsemigroup P_{dc} generated by $(\overline{d,c})$ and P.
Then

$$P_{cd} \supsetneq P \supseteq N$$

and

$$P_{dc} \supsetneq P \supseteq N$$

since neither $(\overline{c,d})$ nor $(\overline{d,c})$ are contained in P. This contradicts the maximality of P unless $(\overline{0,0}) \varepsilon P_{cd}$ and $(0,0) \varepsilon P_{dc}$. This means

$$(\overline{u,v}) * m(\overline{c,d}) = (\overline{0,0})$$

and

$$(\overline{w,x}) * n(\overline{d,c}) = (\overline{0,0})$$

for $(\overline{u,v})$ and $(\overline{w,x})$ in P and some natural numbers n and m.
Then

$$n(\overline{u,v}) * nm\overline{(c,d)} = (\overline{0,0})$$

and

$$m(\overline{w,x}) * mn(\overline{d,c}) = (\overline{0,0}) ,$$

which, upon adding the two equations, yields

$$n(\overline{u,v}) * m\ (\overline{w,x}) = (\overline{0,0}) .$$

This is a contradiction of the closure of P under * since $n(\overline{u,v})$ and $m(\overline{w,x})$ are in P but their sum $(\overline{0,0})$ is not. Therefore, at least one of $(\overline{c,d}) \varepsilon P$ or $(\overline{d,c}) \varepsilon P$ holds.

This completes the proof that exactly one of $(\overline{c,d}) = (\overline{0,0})$, or $(\overline{c,d}) \varepsilon P$ or $(\overline{d,c}) \varepsilon P$ holds.

For any $(\overline{a,b})$ and $(\overline{c,d})$ in G , define a relation $<$ such that

$$(\overline{a,b}) < (\overline{c,d}) \text{ iff } (\overline{a,b}) * (\overline{d,c}) \varepsilon P .$$

Define a relation $\leqq$ on G by

$$(\overline{a,b}) \leqq (\overline{c,d}) \text{ iff } (\overline{a,b}) < (\overline{c,d}) \text{ or } (\overline{a,b}) = (c,d),$$

where $(\overline{a,b})$, $(\overline{c,d}) \varepsilon G$.

The relation $\leqq$ is reflexive since $(\overline{a,b}) = (\overline{a,b})$.

The relation $\leqq$ is antisymmetric since if $(\overline{a,b}) \leqq (\overline{c,d})$ and $(\overline{c,d}) \leqq (\overline{a,b})$, then $(\overline{a,b}) = (\overline{c,d})$. For if $(a,b) \neq (\overline{c,d})$, then $(\overline{a,b}) * (\overline{d,c}) \varepsilon P$ and $(\overline{c,d}) * (\overline{b,a}) \varepsilon P$. Then their sum $(\overline{0,0})$ would have to be in P, which is a contradiction. Therefore, $(\overline{a,b}) = (\overline{c,d})$.

The relation $<$ is transitive since if $(\overline{a,b}) < (\overline{c,d})$ and $(\overline{c,d}) < (\overline{e,f})$, then $(\overline{a,b}) * (\overline{d,c}) \varepsilon P$ and $(\overline{c,d}) * (\overline{f,e}) \varepsilon P$. Hence, $(\overline{a,b}) * (\overline{f,e}) \varepsilon P$. This implies that $(\overline{a,b}) < (\overline{e,f})$. Now it is easily seen that $\leqq$ is transitive.

For any two elements $(\overline{a,b})$, $(\overline{c,d}) \varepsilon G$, either $(\overline{a,b}) < (\overline{c,d})$ or $(\overline{c,d}) < (\overline{a,b})$ or $(\overline{a,b}) = (\overline{c,d})$. Assume $(\overline{a,b}) < (\overline{c,d})$ and $(\overline{a,b}) = (\overline{c,d})$ do not hold. Since $(\overline{a+d,b+c}) \neq (\overline{0,0})$, and $(\overline{a+d,b+c}) \notin P$, $(\overline{b+c,\ a+d})$ must be in P. This implies that $(\overline{b,a}) * (\overline{c,d}) \in P$. Hence $(\overline{c,d}) < (\overline{a,b})$. A similar proof follows for the other combinations.

This completes the proof that the relation $\leqq$ is a total ordering of G.

Now consider the group ring T_o of G with respect to an integral domain R. Let $T_{\overline{S}}$ be the subsemigroup ring of T_o where $\overline{S}$ is the isomorphic image of S in G, as discussed in theorem 4-9. Then the semigroup ring T of S with respect to R is isomorphic to $T_{\overline{S}}$ under the isomorphism Φ defined by

$$\Phi : T \to T_{\overline{S}}$$
$$\Phi(f) = \bar{f}$$

where $\bar{f}(\bar{s}) = f(s)$ and $\bar{s} = (\overline{s,0})$. Notice the range elements, $\{f_s\}_{s \in S}$ and $\{f_{(\overline{s,0})}\}_{(\overline{s,0}) \in \overline{S}}$, are identical.

<u>Theorem 4-12</u>. The group ring T_o of G with respect to R is an integral domain.

<u>Proof</u>. By theorem 4-4, T_o is a commutative ring since both G and R are commutative.

By theorem 4-5, T_o is a ring with identity since G and R have identities.

Now, we need only show that T_o contains no divisors of zero to complete the proof.

Assume $f \neq \theta$ is a zero divisor in T_o .

Then there exists a nonzero element $g \varepsilon T_o$ such that $f \odot g = \theta$.

Since elements of T_o are isomorphic to polynomials in an indeterminant x, with coefficients in R and exponents in G, we can consider the associated polynomials of g and f defined as

$$f \leftrightarrow \sum_{s \varepsilon G} f(s)\, x^s$$

and

$$g \leftrightarrow \sum_{s \varepsilon G} g(s)\, x^s \quad .$$

Multiplication and addition in T_o correspond to the same operations on the associated polynomials.

Since G is totally ordered and each of the functions is finitely non-zero, m will be considered the degree of the polynomial $\sum_{s \varepsilon G} f(s)x^s$ if m is the largest element of G such that $f(m)x^m \neq 0$.

Then $f \odot g = \theta$ means

$$\sum_{s \varepsilon G} f(s)\, x^s \cdot \sum_{s \varepsilon G} g(s)\, x^s = 0$$

Thus $f(m) \cdot g(n)$ must be 0, where m and n are the degrees of $\sum_{s \varepsilon S} f(s)x^s$ and $\sum_{s \varepsilon S} g(s)x^s$ respectively. This contradicts that R is free of zero divisors since f(m) and g(n) are non-zero elements in the integral domain R. Therefore, T_o must be free of zero divisors.

This completes the proof that T_o is an integral domain.

Proof of theorem 4-7. The essentials of the first part of this proof have been given in the previous theorems. All that remains is to note that the semigroup ring T of S with respect to R is isomorphic to $T_{\bar{S}}$, a subring of T_o. Then T_o an integral domain implies $T_{\bar{S}}$ is an integral domain. Therefore, T is an integral domain.

Assume the semigroup ring T of S with respect to R is an integral domain. We will show that if properties (1), (2) or (3) fail, then T will not be an integral domain.

(1) Assume R has a zero divisor, i.e., $a\cdot b=0$ with $a\neq 0$ and $b\neq 0$ for some $a,b\in R$. Then consider $f,g\in T$ defined by

$$f(s) = \begin{cases} a \text{ if } & s=u_o , \\ 0 \text{ if } & s\neq_o , \end{cases}$$

$$g(s) = \begin{cases} b \text{ if } & s=v_o , \\ 0 \text{ if } & s\neq v_o , \end{cases}$$

where u_o,v_o are fixed elements in S. Let the product $u_o\cdot v_o$ be s_o . Then

$$\begin{aligned} (f \odot g)(s_o) &= \sum_{uv=s_o} f(u)\cdot g(v) \\ &= 0 + f(u_o)\cdot g(v_o) \\ &= a\cdot b \\ &= 0 . \end{aligned}$$

Hence, $f \odot g = \Theta$ where neither f nor g is Θ . Hence, T has divisors of zero and is not an integral domain.

(2) Assume S is not cancellative. Then for some $a,b,c \varepsilon S$, $a+c = b+c$ and $a \neq b$. Consider the associated polynomials,

x^{a+c} and x^{b+c} , of two elements of T . Then

$$x^{a+c} = x^{b+c}$$

$$x^{a+c}-x^{b+c} = 0$$

$$x^a x^c - x^b x^c = 0$$

$$(x^a - x^b)x^c = 0$$

where neither factor is zero. Hence, T is not an integral domain.

(3) Assume n is the smallest natural number such that $ns \neq nt$ for distinct elements $s,t \varepsilon S$. Then

$$0 = x^{ns} - x^{nt}$$

$$= (x^s - x^t)(x^{(n-1)s} + x^{(n-2)s+t}$$

$$+ \ldots + x^{s+(n-2)t} + x^{(n-1)t}) .$$

But $x^s - x^t \neq 0$ and, since n is minimal and positive, the exponents of the terms in the right factor are distinct. For assume any two terms have equal exponents. Then

$$(n-i)s + (n-j)t = (n-p)s + (n-q)t .$$

where

$$(n-i) + (n-j) = (n-p) + (n-q) = n-1 .$$

Therefore,

$$(p-i)s = (j-q)t.$$

But

$$(n-i) + (n-j) = (n-p) + (n-q) = n-1$$

implies

$$p-i = j-q ,$$

and

$$p+q = i+j = n+1 .$$

Then

$$0 < p + q < n \quad \text{and} \quad 0 < i + j < n .$$

Hence,

$$p < n \quad \text{and} \quad j < n .$$

Therefore,

$$p-i < p < n \quad \text{and} \quad j-q < j < n .$$

This is a contradiction of the minimality of n since we now have

$$(p-i)s = (j-q)t$$

where

$$p-i = j-q < n .$$

Thus T has divisors of zero and therefore is not an integral domain.

This completes the proof of theorem 4-7.

CHAPTER V

THE COMPUTER PROGRAM FOR GENERATING SEMIGROUP RINGS

The program for generating semigroup rings is written in the Symbolic Programming System (SPS) for the IBM 1620 computer. Limitations on the order of the semigroup rings that it will generate are due to memory size and not the technique of the program. The program could easily be adapted for larger orders. It can be divided into three somewhat disjoint parts, the listing of the functions, the additive structure, and the multiplicative structure. Both the semigroup and the ring will be represented by non-negative integers.

I. THE LISTING OF FUNCTIONS

The set of functions is the same for any fixed order semigroup and ring. The number of elements in any finite semigroup ring is n^r where n is the order of the ring and r is the order of the semigroup.

These functions are stored in blocks consisting of two parts. The first part is the number of the function and the second contains the functional values of the functions beginning with $f(0)$ on the far right of each block and proceeding to the left to $f(n)$ where n is the order of the semigroup.

These functions are generated by repeated addition of

1 modulo N where N is the order of the ring. The functions are numbered as they are generated and it is this number that is punched as output.

For example, the functions for a semigroup ring whose underlying semigroup is of order two and ring is of order three are found in storage as follows:

$\bar{0}$	0	$\bar{0}$	0
$\bar{0}$	1	$\bar{0}$	1
$\bar{0}$	2	$\bar{0}$	2
$\bar{0}$	3	$\bar{1}$	0
$\bar{0}$	4	$\bar{1}$	1
$\bar{0}$	5	$\bar{1}$	2
$\bar{0}$	6	$\bar{2}$	0
$\bar{0}$	7	$\bar{2}$	1
$\bar{0}$	8	$\bar{2}$	2

where the first two digits in each block are the numbers of the functions and the second are the functions.

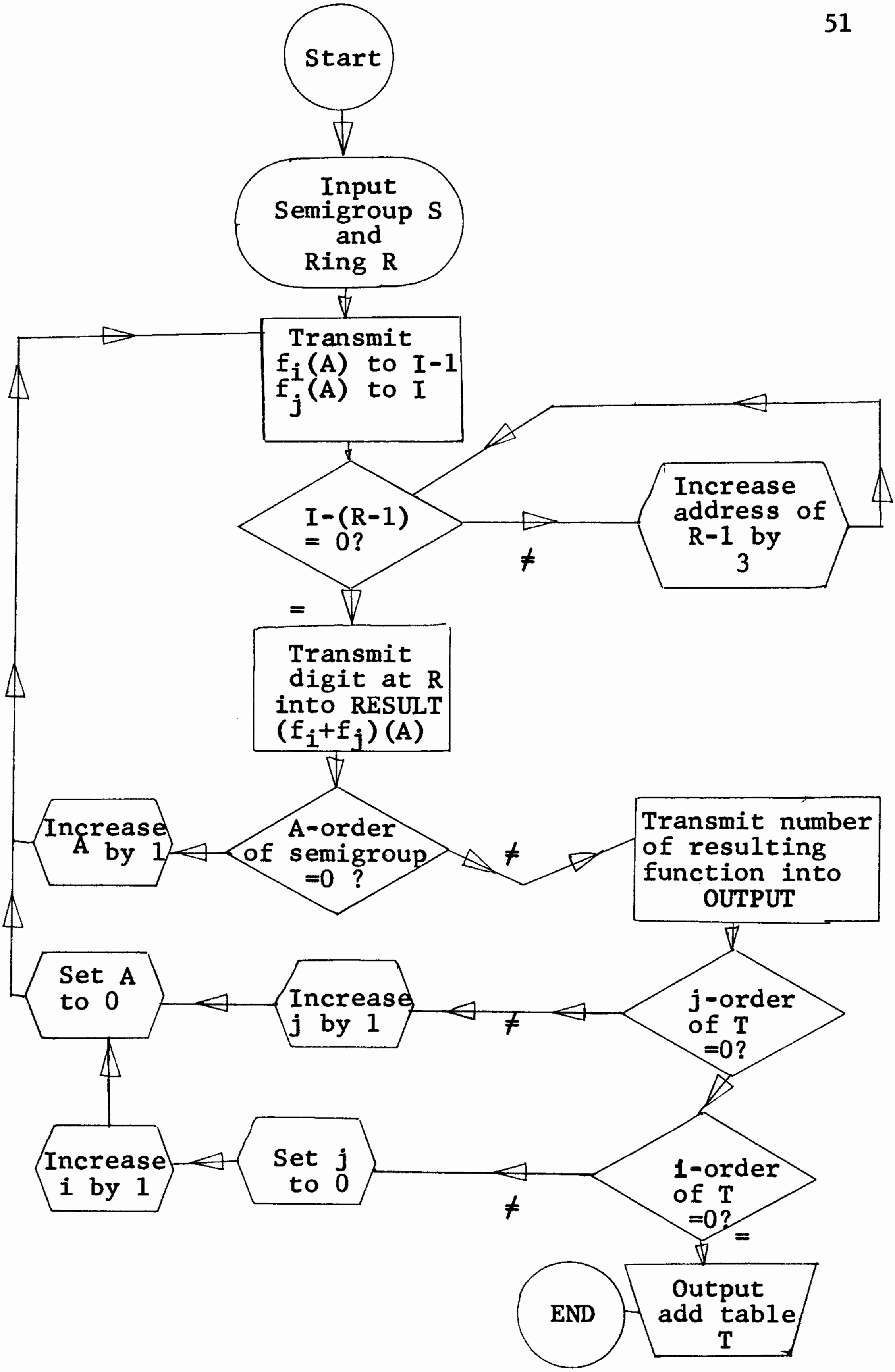
II. THE ADDITIVE STRUCTURE

To generate the addition table for a semigroup ring, the semigroup and the ring are entered into storage as blocks of three digits $\bar{a}$ b c where $a \circ b = c$ and $\circ$ is an operation of the semigroup or ring.

Since addition is pointwise, to add two functions the functional values at each element of the semigroup must be found. To add f_i and f_j , $f_i(0)$ and $f_j(0)$ are transmitted into a two digit field I. Then the addition table for the ring is searched by comparing the first two digits in each block and I. When they match, the third digit (the sum) is stored in the result area. This is repeated for each element in the semigroup. When the sum at each point is computed, the result is compared with the listing of functions and the number of the resulting function is stored in the output area.

This procedure is repeated by fixing i and increasing j by 1 until j equals the order of the semigroup ring. Then j is set back to zero and i is increased by 1. This process continues until i equals the order of the semigroup ring. This has the effect of proceeding across the table, down one row, and across again.

A flow chart for the program generating the addition table for the semigroup ring is shown below.


Start
Input Semigroup S and Ring R
Transmit $f_i(A)$ to I-1 $f_j(A)$ to I
I-(R-1) = 0?
≠
Increase address of R-1 by 3
=
Transmit digit at R into RESULT $(f_i+f_j)(A)$
A-order of semigroup =0 ?
≠
Increase A by 1
Transmit number of resulting function into OUTPUT
j-order of T =0?
≠
Increase j by 1
Set A to 0
i-order of T =0?
≠
=
Set j to 0
Increase i by 1
Output add table T
END

III. THE MULTIPLICATIVE STRUCTURE

As multiplication in the semigroup ring is convolution, the multiplicative portion of the program is quite complex.

The program begins by transferring to address of the two functions f_i, f_j to be multiplied into A1 and A2. The addition table of the semigroup is searched to find the factors of $0,1,2,\ldots,$ in succession.

The addresses in A1 and A2 refer to the right hand end of the blocks representing the two functions, i.e., $f_i(0)$ and $f_j(0)$. The multiplication of two functions acts from right to left on the functional values beginning with $f(0)$.

For $a,b \in S$ factors of zero, a is subtracted from the address in A1 yielding the address of $f_i(a)$, and b from A2 yielding $f_j(b)$. The two digits, $f_j(a)$ and $f_j(b)$, are transferred into RESULT 1 to be multiplied.

The multiplication is performed by comparing RESULT 1 and the first two digits in the blocks representing the multiplication table of the ring. When they match, the third digit (the product) is transferred into RESULT 2-1. The functions are then evaluated at the next pair of factors of zero. These digits are multiplied as before and their product is stored in RESULT 2.

The sum of the two digits in RESULT 2 is then found by searching the addition table of the ring. This sum is transferred into RESULT 2.

The above procedure is repeated until all of the factors of zero have been used. Then the final entry in RESULT 2 is transmitted into RESULT.

This entire process is repeatdd for 1,2,... until all the elements in the semigroup have been used. RESULT contains $(f_i \odot f_j)(0)$, RESULT - 1 contains $(f_i \odot f_j)(1)$, ... The final product $f_i \odot f_j$ is stored in RESULT. The number of the resulting product is found as before and transmitted into the output area.

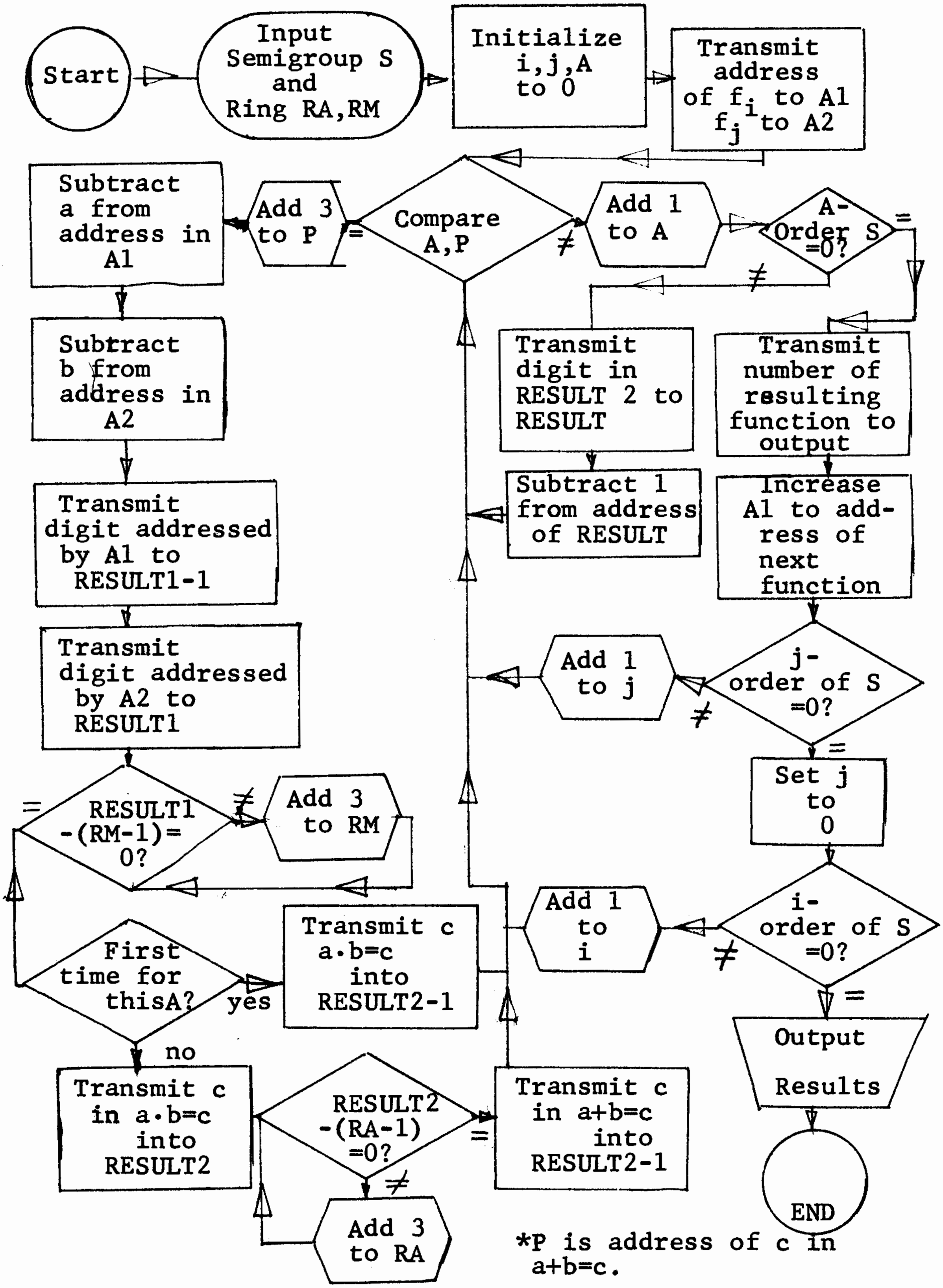
This procedure for multiplication has the effect of proceeding across the table, down one row, and across again.

Since the number of digits required to represent a row of either table of the semigroup ring is greater than the punching positions available on the card(80) , the output cards must be sorted so as to list the table properly. The resulting format of the output is illustrated by the following diagram:

.	$f_1, \ldots, f_{80}, f_{81}, \ldots, f_{160}, \ldots$		
f_1	1st card	2nd card	...
.			
.			
.			
f_{80}			
.	Page One	Page Two	...
.			
.			
f_{160}			
.			
.			
.			

.

A flow chart explanation of the multiplicative structure of the semigroup ring is shown below.



*P is address of c in a+b=c.

CHAPTER VI

SUMMARY AND RECOMMENDATIONS

This thesis as a study of semigroup rings has shed some light on the relationship betwen the structure of a semigroup ring and the structures of the underlying semigroup and ring.

The more important relationships examined include necessary and sufficient conditions for a semigroup ring to be commutative, to be a zero ring, or to be an integral domain as well as various conditions on S, R or T which yield information about the presence of identities in S and R when T has an identity.

In theorem 4-6, it seemed necessary from examination of several examples to require that the semigroup S be cancellative in order to show that the semigroup ring having an identity implies that both the semigroup and the ring have identities. Here an unanswered question may be raised of whether or not an example can be constructed of a semigroup ring with identity in which the underlying semigroup is not cancellative.

BIBLIOGRAPHY

1. Clifford, A. H. and G. B. Preston. _The Algebraic Theory of Semigroups_. Vol. 1 . Providence: American Mathematical Society, 1961. 224 pp.

2. Coleman, D. B. "Idempotents in Group Rings" . _The Proceedings of the American Mathematical Society_. Vol. 17, No. 4, August 1966. pp 962.

3. _______. "On the Modular Group Ring of a p-Group" . _The Proceedings of the American Mathematical Society_. Vol. 15, No. 4, August 1964 . pp 511.

4. Gilmer, Robert . "A Note on Semigroup Rings" . _Notices of the American Mathematical Society_, Vol. 14, No. 7, Issue No. 101, November 1967. pp 21.

5. Hall, Marshall Jr. _The Theory of Groups_. New York: The MacMillan Co., 1959. 431 pp.

6. Jacobsen, Nathan. _Lectures in Abstract Algebra_. Vol. 1, Princeton, New Jersey: D. VanNostrand Company, Inc., 1951. pp 92-95.

7. Lambeck, Joachin. _Lectures on Rings and Modules_. Waltham, Massachusetts, Toronto, and London: Blaisdell Publishing Co., 1966. 183 pp.

8. McCoy, Neal H. _Rings and Ideals_. Buffalo, New York: The Mathematical Association of America, 1948. 216 pp.

9. Redei, L. _Algebra_. Vol. 1 . Oxford, London, Edinburgh, New York, Toronto, Sydney, Paris, Brauschweig: Pergamon Press, 1967. 823 pp.

10. Schneider, Hans and Julian Weissglass. "Group Rings, Semigroup Rings and Their Radicals" . _The Journal of Algebra_. Vol. 5, No. 1, January 1967. pp 1-15.

```
       DC   3,100
       DC   3,111
       DC   3,122
       DC   3,133
       DC   3,200
       DC   3,210
       DC   3,220
       DC   3,230
       DC   3,300
       DC   3,311
       DC   3,322
       DC   3,333
S3XR4  DS   5
       DS   320
       DC   30,000000000000000000000000000000
OUTPA  DC   50,00000000000000000000000000000000000000000000000000
       DC   30,000000000000000000000000000000
OUTPB  DC   50,00000000000000000000000000000000000000000000000000
       DC   30,000000000000000000000000000000
OUTPC  DC   50,00000000000000000000000000000000000000000000000000
A1     DC   5,00000
A2     DC   5,00000
B1     DC   5,00000
B2     DC   5,00000
B3     DC   5,00000
D1     DC   2,00
D2     DC   2,00
FA1    DC   5,00000
FA2    DC   5,00000
FACT   DC   2,00
1      DC   2,00
L      DC   4,0004
LL     DC   2,00
M      DC   4,0019
N      DC   4,0000
O      DC   4,0079
P      DC   2,00
Q      DC   5,00000
R      DC   2,00
S      DC   5,00000
S1     DC   2,00
U      DC   5,00000
U1     DC   2,00
V      DC   2,00
W      DC   5,00000
BL     DNB  2
F1XF2  DC   2,00
```

APPENDIX

I. A PRINTOUT OF A COMPUTER PROGRAM

The following is a listing of the program used to generate examples of semigroup rings. It is written in the symbolic programming system (SPS) for the IBM 1620 computer.

Also included in this section is one example of the resulting semigroup ring as output by the computer.

```
        DORG402
S3      DC   3,000
        DC   3,010
        DC   3,020
        DC   3,100
        DC   3,110
        DC   3,200
        DC   3,121
        DC   3,211
        DC   3,222
        DC   3,555
R1A     DC   3,000
        DC   3,011
        DC   3,022
        DC   3,033
        DC   3,101
        DC   3,110
        DC   3,123
        DC   3,132
        DC   3,202
        DC   3,213
        DC   3,220
        DC   3,231
        DC   3,303
        DC   3,312
        DC   3,321
        DC   3,330
R1M     DC   3,000
        DC   3,010
        DC   3,020
        DC   3,030
        DC   3,100
        DC   3,111
```

```
A1XA2 DC  2,00
RES   DC  3,000
RESULTDC  3,000
CNTER DC  2,00
LIST  DAC 37,THE LISTING OF FUNCTIONS IS COMPLETE'
INST  DAC 40,ENTER THE ORDER OF THE SEMIGROUP RING R'
INST1 DAC 48,THIS COMPLETES THE ADD TABLE FOR ALL SEMIGROUP'
INST2 DAC 38,RINGS OF ORDER R WITH THIS FIXED RING'
INST3 DAC 49,THIS COMPLETES THE MULTIPLICATION TABLE FOR THIS'
INST4 DAC 16, SEMIGROUP RING'
INST5 DAC 49,SET SWITCH ONE ON IF NEW ADD TABLE IS NOT NEEDED'
INST6 DAC 22,ENTER A NEW SEMIGROUP'
INST7 DAC 47,SWITCH TWO ON IF USING NEW MULT TABLE FOR RING'
INST8 DAC 25,ENTER NEW TABLE FOR RING'
INST9 DAC 48,SET SWITCH THREE ON IF ENTERING A NEW SEMIGROUP'
INST10DAC 50,SWITCH FOUR ON IF ENTERING NEW ADD TABLE FOR RING'
INPUTADSS 80
INPUTBDSS 80
ZERO  DC  5,00000
ZER   DC  2,00
START DC  5,00000
TRANS IF  S3XR4,START
      SF  S3XR4-2
      AM  *-18,5,10
      AM  *-18,5,10
      AM  START,1,10
      AM  START-3,1,10
      AM  N,1,10
      C   N,L
      BN  TRANS
      SM  START,4,10
      AM  START,10,10
      AM  L,4,10
      C   L,M
      BN  TRANS
      SM  START,40,10
      AM  START,100,9
      AM  M,16,10
      CM  N,64,8
      BN  TRANS
      RCTY
      WATYLIST
      RCTY
      WATYINST
      RNTYR-1
      TDM OUTPA-79,1
      TDM OUTPB-79,2
```

```
       TDM OUTPC-79,3
       TD  OUTPA-75,BL
       TD  OUTPA-76,BL
       TD  OUTPB-75,BL
       TD  OUTPB-76,BL
       TD  OUTPC-75,BL
       TD  OUTPC-76,BL
       SF  OUTPA-78
       SF  OUTPB-78
       SF  OUTPC-78
       CF  OUTPA-79
       CF  OUTPB-79
       CF  OUTPC-79
       CF  OUTPA-49
       CF  OUTPB-49
       CF  OUTPC-49
       RCTY
       WATYINST5
       H
       BC1 STAR1
ST     TD  I-1,S3XR4,27
       SF  I-1
       SM  *-13,1,7
ST2    TD  I,S3XR4,27
       SM  *-1,1,7
       CF  I
COMP   C   R1A-1,I,2
       BE  EQUAL
       AM  *-18,3,7
       AM  W,3,7
       B   COMP
EQUAL  AM  W,R1A,7
       TD  RESULT,W,211
       SM  *-6,1,7
       TF  W,ZERO
       TFM COMP+6,R1A,7
       SM  COMP+6,1,7
       AM  V,1,10
       CM  V,3,10
       BE  C1
       B   ST
C1     AM  ST+11,3,7
       AM  ST2+11,3,7
       SF  RESULT-2
       TFM EQUAL+18,RESULT,7
       SF  RESULT-2
C2     C   S3XR4,RESULT,2
```

```
        BE   END1
        AM   *-18,5,7
        AM   Q,5,7
        B    C2
END1    TFM  C2+6,S3XR4,7
        AM   Q,S3XR4,7
        SM   Q,3,7
        TF   OUTPA-73,Q,211
        CF   OUTPA-74,,2
        AM   *-18,3,7
        AM   *-18,3,7
        TD   OUTPA-72,BL,2
        AM   *-6,3,7
        AM   P,1,10
        CM   P,25,10
        BN   *+60
        AM   END1+42,5,7
        AM   END1+54,5,7
        AM   END1+90,5,7
        TF   P,ZER
        TF   Q,ZERO
        TF   V,ZER
        TF   RESULT,ZERO-2
        TF   W,ZERO
        AM   ST+11,5,7
        AM   U1,1,10
        C    U1,R
        BE   *+24
        B    ST
        TF   P,ZER
        TF   U1,ZER
        TFM  ST+11,S3XR4,7
        AM   ST2+11,5,7
        CF   OUTPA-78
        CF   OUTPB-78
        CF   OUTPC-78
        WNCDOUTPA-79
        WNCDOUTPB-79
        WNCDOUTPC-79
        SF   OUTPA-78
        SF   OUTPB-78
        SF   OUTPC-78
        AM   OUTPA-77,1,10
        AM   OUTPB-77,1,10
        AM   OUTPC-77,1,10
        TFM  END1+42,OUTPA-73,7
        TFM  END1+54,OUTPA-74,7
        TFM  END1+90,OUTPA-72,7
```

```
      AM   S1,1,10
      C    S1,R
      BE   *+24
      B    ST
      RCTY
      WATYINST1
      WATYINST2
      TF   S1,ZER
      TFM  ST2+11,S3XR4,7
      TF   OUTPA-77,ZER
      TF   OUTPB-77,ZER
      TF   OUTPC-77,ZER
STAR1 TFM  A2,S3XR4,7
      SF   A2-4
      TFM  A1,S3XR4,7
      SF   A1-4
STAR2 TD   FACT,S3,7
      SF   FACT-1
      C    FACT,LL
      BE   EN1
      TD   RES,FIXF2-1,27
      SM   *-6,1,7
      TFM  TRA1+6,FIXF2-1,7
      TF   CNTER,ZER
      AM   LL,1,10
      CM   LL,3,10
      BE   EN2
      B    STAR2+12
EN1   SM   STAR2+11,1,7
      TD   FA2,STAR2+11,11
      CF   FA2
      SM   STAR2+11,1,7
      TD   FA1,STAR2+11,11
      CF   FA1
      SF   FA2-4
      SF   FA1-4
      AM   STAR2+11,2,7
      S    A2,FA2
      S    A1,FA1
      TD   A1XA2-1,A1,11
      TD   A1XA2,A2,11
      SF   A1XA2-1
      CF   A1XA2
COM1  C    RIM-1,A1XA2,2
      BE   *+48
      AM   B1,3,7
      AM   COM1+6,3,7
      B    COM1
```

```
        AM   B1,RIM,7
TRA1    TD   FIXF2-1,B1,211
        CF   FIXF2
        AM   CNTER,1,10
        CM   CNTER,2,10
        BN   ENO
        SF   FIXF2-1
COM2    C    RIA-1,FIXF2,2
        BE   *+48
        AM   COM2+6,3,7
        AM   B2,3,7
        B    COM2
        AM   B2,RIA,7
        TD   FIXF2-1,B2,11
        B    *+24
ENO     AM   TRA1+6,1,7
        AM   STAR2+11,3,7
        TF   B1,ZERO
        TF   B2,ZERO
        TFM  COM1+6,RIM-1,7
        TFM  COM2+6,RIA-1,7
        B    STAR1
EN2     SF   RES-2
        C    S3XR4,RES,2
        BE   EN3
        AM   *-18,5,7
        AM   B3,5,7
        B    EN2+12
EN3     AM   B3,S3XR4,7
        SM   B3,3,7
        TF   OUTPA-73,B3,211
        CF   OUTPA-74,,2
        AM   *-18,3,7
        AM   *-18,3,7
        TD   OUTPA-72,BL,2
        AM   *-6,3,7
        AM   P,1,10
        CM   P,25,10
        BN   *+60
        AM   EN3+30,5,7
        AM   EN3+42,5,7
        AM   EN3+78,5,7
        TF   P,ZER
        TF   RES,ZERO-2
        TFM  STAR2+11,S3,7
        TFM  STAR2+54,RES,7
        TF   LL,ZER
        TF   FACT,ZER
```

```
TFM COM1+6,RIM-1,7
TFM COM2+6,RIA-1,7
TF  B1,ZERO
TF  B2,ZERO
TF  B3,ZERO
TFM EN2+18,S3XR4,7
TF  FIXF2,ZER
TF  A1XA2,ZER
TF  A1,ZERO
TF  A2,ZERO
TF  FA1,ZERO
TF  FA2,ZERO
AM  D1,1,10
C   D1,R
BE  *+36
AM  STAR1+11,5,7
B   STAR1
TF  D1,ZER
CF  OUTPA-78
CF  OUTPB-78
CF  OUTPC-78
WNCDOUTPA-79
WNCDOUTPB-79
WNCDOUTPC-79
SF  OUTPA-78
SF  OUTPB-78
SF  OUTPC-78
TFM EN3+30,OUTPA-73,7
TFM EN3+42,OUTPA-74,7
TFM EN3+78,OUTPA-72,7
AM  OUTPA-77,1,10
AM  OUTPB-77,1,10
AM  OUTPC-77,1,10
AM  STAR1+35,5,7
TFM STAR1+11,S3XR4,7
TF  P,ZER
AM  D2,1,10
C   D2,R
BE  *+24
B   STAR1
RCTY
WATYINST3
WATYINST4
H
TF  OUTPA-77,ZER
TF  OUTPB-77,ZER
TF  OUTPC-77,ZER
TF  D2,ZER
```

```
      TFM STAR1+35,S3XR4,7
SETUP RCTY
      WATYINST9
      H
      BC3 *+24
      B   *+72
      RCTY
      WATYINST6
      RNCDINPUTA
      SF  INPUTA
      TF  S3+24,INPUTA+26
      RCTY
      WATYINST10
      H
      BC4 *+24
      B   *+72
      RCTY
      WATYINST8
      RNCDINPUTB
      SF  INPUTB
      TF  RIA+45,INPUTB+47
      RCTY
      WATYINST7
      H
      BC2 *+24
      B   FLAG
      RCTY
      WATYINST8
      RNCDINPUTB
      SF  INPUTB
      TF  RIM+45,INPUTB+47
FLAG  SF  S3-2,,2
      AM  *-6,3,7
      CM  FLAG+6,00527,7
      BN  FLAG
      TFM FLAG+6,S3,7
      B   TRANS
      DENDSETUP
```

II. AN EXAMPLE OF THE OUTPUT

Example A-1. An example of the computed semigroup ring appears in the following two pages. The semigroup used is given by the following Cayley table:

·	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

The additive and multiplicative structures for the ring used are given by the following Cayley tables:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	0	0	0
3	0	1	2	3

.

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