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An Investigation of Mikusinski's Operational Calculus

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CHAPTER I

THE DEVELOPMENT OF THE FIELD OF MIKUSINSKI OPERATORS

Let F be the set of all complex-valued functions $f = f(t)$, $g = g(t)$, ... defined for $0 \leq t < \infty$, each of which possess continuous derivatives. We only require the existence of the right-hand derivative at zero.

Definition. Two functions f(t) and g(t) of F are said to be equal if $f(t) = g(t)$ for $0 \leq t < \infty$.

Addition and multiplication will be defined in such a way that F forms a ring. Addition of two elements f(t) and g(t) in F is defined pointwise. It is easily shown that the commutative and associative laws hold for addition. Define multiplication in F by the so-called Duhamel integral (1:61)

$$fg = \frac{d}{dt}\int_0^t f(\tau)g(t-\tau)d\tau \tag{1.1}$$

and call fg the function product of f and g. If a differentiable function is written without an argument, it is always understood to be an element of F. We shall indicate functional multiplication by a dot, i.e., we use the notation $fg = f(t) \cdot g(t)$. In ambiguous cases, or when we want to emphasize function product we write $fg = f \cdot g$. The usual or pointwise product of two functions f(t) and g(t) will be called the value product and will be written without a dot, i.e., f(t)g(t).

In the case of constant functions a, b observe that

$$a \cdot b = \frac{d}{dt}\int_0^t abdt = ab\,\frac{d}{dt}\int_0^t d\tau = ab.$$

Hence, for constants the value product and the function product are equal and we can omit the dot.

Now let f(t) be an arbitrary function in F and let a be a constant. Then we have

$$a \cdot f(t) = \frac{d}{dt}\int_0^t af(\tau)d\tau = af(t).$$

Here again, we see that the value product and the function product are equal and we may omit the dot. Also we see that F has a unit element for function multiplication, namely a = 1.

Notice, however, that the value product and the functional product are not always equal. For example,

$$e^{\alpha t} \cdot e^{\alpha t} = \frac{d}{dt}\int_0^t e^{\alpha\tau}\, e^{\alpha(t-\tau)}d\tau$$

$$= \frac{d}{dt}\int_0^t e^{\alpha t}\, d\tau$$

$$= \frac{d}{dt}\,(e^{\alpha t}t)$$

$$= e^{\alpha t}(1 + \alpha t). \tag{1.2}$$

To establish the commutative law for functional multiplication, let $\tau = t - x$, $d\tau = -dx$. Then

$$fg = \frac{d}{dt}\int_0^t f(\tau)g(t - \tau)d\tau$$

$$= \frac{-d}{dt}\int_t^0 f(t - x)g(x)dx$$

$$= \frac{d}{dt}\int_0^t g(x)f(t - x)dx$$

$$= gf.$$

To show that F is closed under functional multiplication we first write

$$fg = \frac{d}{dt}\int_0^t f(\tau)g(t - \tau)d\tau$$

and carry out the differentiation to obtain

$$fg = \int_0^t f(\tau)g'(t - \tau)d\tau + f(t)g(0).$$

Now we differentiate both sides with respect to t and obtain

$$(fg)' = \frac{d}{dt}\int_0^t f(\tau)g'(t-\tau)d\tau + f'(t)g(0).$$

Letting $x = t - \tau$, $dx = -d\tau$, we have

$$(fg)' = -\frac{d}{dt}\int_t^0 f(t-x)g'(x)dx + f'(t)g(0)$$

$$= \frac{d}{dt}\int_0^t g'(x)f(t-x)dx + f'(t)g(0)$$

$$= \int_0^t g'(x)f'(t-x)dx + g'(t)f(0) + f'(t)g(0)$$

$$= \int_0^t g'(\tau)f'(t-\tau)d\tau + g'(t)f(0) + f'(t)g(0).$$

Since the latter integrand is continuous, the integral is a continuous function of its upper limit and we have that $fg\varepsilon F$, for the remaining two terms on the right-hand side are clearly continuous.

In proving the associative law for multiplication we make use of Dirichlet's formula,

$$\int_0^z\int_0^y F(x,y)dxdy = \int_0^z\int_x^z F(x,y)dydx.$$

Applying Dirichlet's formula, we have

$$(fg)h = \frac{d}{dt}\int_0^t \left(\frac{d}{dx}\int_0^x f(\tau)g(x-\tau)d\tau\right) h(t-x)dx$$

$$= \frac{d}{dt}\int_0^t \left(\int_0^x f(\tau)g'(x-\tau)d\tau+f(x)g(0)\right) h(t-x)dx$$

$$= \frac{d}{dt}\int_0^t\int_0^x f(\tau)\ g'(x-\tau)d\tau h(t-x)dx$$

$$+ \frac{d}{dt}\int_0^t f(x)g(0)h(t-x)dx$$

$$= \frac{d}{dt}\int_0^t f(\tau)\ \int_\tau^t g'(x-\tau)h(t-x)dxd\tau$$

$$+ \frac{d}{dt}\int_0^t f(x)g(0)h(t-x)dx.$$

Substituting $t-x = y$, $dx = -dy$, and $\tau = t-\sigma$, $d\tau = -d\sigma$, it follows that

$$(fg)h = \frac{d}{dt}\int_t^0 f(t-\sigma)\ \int_{t-\tau}^0 g'(\sigma-y)h(y)(-dy)(-d\sigma)$$

$$+ \frac{d}{dt}\int_t^0 f(t-y)g(0)h(y)(-dy)$$

$$= \frac{d}{dt}\int_0^t f(t-\sigma)\ \int_0^\sigma g'(\sigma-y)h(y)dyd\sigma$$

$$+ \frac{d}{dt}\int_0^t f(t-\sigma)g(0)h(\sigma)d\sigma$$

$$= \frac{d}{dt}\int_0^t \left(\int_0^\sigma g'(\sigma-y)h(y)dy + g(0)h(\sigma)\right) f(t-\sigma)d\sigma$$

$$= \frac{d}{dt}\int_0^t \left(\frac{d}{d\sigma}\int_0^\sigma h(y)g(\sigma-y)dy\right) f(t-\sigma)d\sigma$$

$$= (gh)f$$

$$= f(gh).$$

The distributive law follows easily.

$$f\{g+h\} = \frac{d}{dt}\int_0^t f(t-\tau)\{g(\tau) + h(\tau)\}d\tau$$

$$= \frac{d}{dt}\int_0^t f(t-\tau)g(\tau)d\tau + \frac{d}{dt}\int_0^t f(t-\tau)h(\tau)d\tau$$

$$= fg + fh.$$

From our discussion this far F is a ring under pointwise addition and functional multiplication.

Before continuing the development of the structure of F, let us investigate some of the properties of functional multiplication needed later.

If $g(t) = t$, then

$$t\cdot f(t) = \frac{d}{dt}\int_0^t f(\tau)(t-\tau)d\tau$$

$$= \int_0^t f(\tau)d\tau,$$

so that functional multiplication of $f(t)$ by t results in integration of f. In particular, we obtain

$$t \cdot fg = \int_0^t f(\tau)g(t-\tau)d\tau.$$

This integral is called the convolution of the functions f and g.

Let us use a dot in front of the exponent to denote powers with respect to the function product, i.e., $f^n = f^{\cdot n}(t)$, and define $t^{\cdot 1} = t$, $t^{\cdot 0} = 1$. By induction we establish the formula

$$t^{\cdot n} = \frac{1}{n!} t^n \tag{1.3}$$

for any positive integer n. For n = 2, we have

$$t \cdot t = \int_0^t \tau d\tau = \frac{t^2}{2} = \frac{t^2}{2!}.$$

If $t^{\cdot k} = \frac{1}{k!} t^k$, then

$$t \cdot t^{\cdot k} = t \cdot \frac{1}{k!} t^k = \frac{1}{k!} \int_0^t \tau^k d\tau = \frac{1}{k!} \cdot \frac{t^{k+1}}{(k+1)} = \frac{1}{(k+1)!} t^{k+1}.$$

Therefore, $t^{\cdot n} = \frac{1}{n!} t^n$ for any positive integer n by induction.

We now establish that F is an integral domain. Since F is a ring we must show that there are no divisors of zero. For this purpose we need Titchmarsh's theorem (1:69): If

$$fg = \frac{d}{dt}\int_0^t f(\tau)g(t-\tau)d\tau = 0$$

and $g \neq 0$, then $f = 0$, i.e., $f(t) = 0$ for all $t \geq 0$. The following proof of this theorem is due to J. Mikusinski (1:72) and is in five parts.

(1) If $f(t) = 0$ in $[0, \alpha]$ and $g(t) = 0$ in $[0, \beta]$, then $fg = 0$ in $[0, \alpha + \beta]$ from the definition of multiplication. For, if $0 \leq \tau \leq t \leq \alpha + \beta$, then either $\tau \leq \alpha$ and $f(\tau) = 0$, or $\tau > \alpha$, i.e., $t - \tau < \beta$, and hence $g(t-\tau) = 0$. The initial intervals in which the functions vanish are additive when multiplying. In particular, since $f = 0$ in $[0, \alpha]$, we have that $f^2 = 0$ in $[0, 2\alpha]$. The converse also holds, i.e., if $f^2 = 0$ in $[0, \alpha]$ then $f = 0$ in $[0, \alpha/2]$ (1:213). This latter result will be used later.

(2) We next associate with every function $f \in F$ a function $f_1 = tf(t)$, where value multiplication by t is understood. The transformation

$$t\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t \tau f(\tau)g(t-\tau)d\tau + \int_0^t f(\tau)(t-\tau)g(t-\tau)d\tau$$

may be written

$$(t\cdot fg)_1 = t\cdot f_1 g + t\cdot fg_1,$$

where multiplication by t is functional multiplication.

(3) Define ε as the greatest non-negative number such that, for given arbitrary f, g ε F with fg = 0 in $\left[0,\ 1\right]$, we have fg = 0 in $\left[0,\ \varepsilon\right]$. The example f(t) = 1 and g(t) = 0 for $0 \leq t \leq 1$, and $g(t) = (t-1)^2$ for $t \geq 1$, shows that $\varepsilon \leq 1$ inasmuch as

$$fg = g_1 = (t-1)^2 t$$

for $t \geq 1$. The existence of ε as defined above is thus ensured. The same ε can also be defined as the greatest number such that, for given f, g ε F with fg = 0 in $\left[0,\ T\right]$, we have $fg_1 = 0$ in $\left[0,\ \varepsilon T\right]$. For, it follows from $f(t)\cdot g(t) = 0$ in $\left[0,\ T\right]$ and the substitution $t = Ts$, $\tau = T\sigma$, $f^*(s) = f(t)$, $g^*(s) = g(t)$, that

$$f(t)\cdot g(t) = \frac{d}{dt}\int_0^t f(\tau)g(t-\tau)d\tau$$

$$= \frac{d}{ds}\int_0^s f^*(\sigma)g^*(s-\sigma)d\sigma = 0$$

provided s lies in the interval $\left[0,\ 1\right]$. If f and g range over the complete set of functions permissible in the second definition of ε, f^* and g^* will range over the set of functions appearing in the first definition. This establishes the equivalence of the two definitions, and the independence of the number ε on the length of the interval T.

(4) Now let f, g be any two functions of F with $fg = 0$ in $[0, T]$. Then $tfg = 0$ and $(fg)_1 = 0$ in the same interval. It thus follows that

$$f_1 g + f g_1 = 0$$

in $[0, T]$. If we multiply this equation by the function fg_1, which vanishes in the interval $[0, \varepsilon T]$ in accordance with the definition of ε in (3), we obtain

$$fgf_1g_1 + (fg_1)^2 = 0$$

in $[0, (1+\varepsilon)T]$, since the interval lengths add up on multiplying as was shown in (1). On the other hand, from (3), $g_1 f = 0$ in $[0, \varepsilon T]$ implies $g_1 f_1 = 0$ in $[0, \varepsilon^2 T]$, and hence $(fg)(f_1g_1) = 0$ in $[0, (1+\varepsilon^2)T]$ by (1). It follows that $(fg_1)^2 = 0$ in $[0, (1+\varepsilon^2)T]$. Hence $fg_1 = 0$ in $[0, \frac{1}{2}(1+\varepsilon^2)T]$ by (1). But f and g are arbitrary functions of F with $fg = 0$ in $[0, T]$. Hence by the definition of ε,

$$\tfrac{1}{2}(1+\varepsilon^2) \leq \varepsilon \Rightarrow (1-\varepsilon^2) \leq 0.$$

But this inequality is only possible for $\varepsilon = 1$. We have thus obtained the important intermediate result that, if $fg = 0$ in $[0, T]$, then

$$fg_1 = \frac{d}{dt} \int_0^t f(t-\tau)g(\tau)\tau d\tau = 0$$

in $[0, T]$.

(5) By employing functional multiplication by t and induction, the above result can be generalized to obtain

$$\int_0^t f(t-\tau)g(\tau)\tau^n d\tau = 0 \qquad (1.4)$$

in $[0, T]$ for $n = 1, 2, 3, \ldots$, provided the relationship holds for $n = 0$. The moment theorem (1:260) states that if $f(x)$ is a continuous function in $[0, a]$, and

$$\int_0^a x^n f(x)dx = 0 \text{ for } n = 0, 1, 2, \ldots,$$

then $f(x) = 0$ for all x in $[0, a]$. Applying the moment theorem to (1.4), we have that

$$f(t-\tau)g(\tau) = 0 \text{ for } 0 \le \tau \le t \le T.$$

If $g(\tau_0) \neq 0$, then we must have $f(t) = 0$ for $0 \le t \le T - \tau_0$. If $fg = 0$ for all t, then we can take T as large as desired, so $f(t) = 0$ for all t.

Titchmarsh's theorem is thus proved and F is an integral domain.

Since F is an integral domain, there is a corresponding quotient field Q. We shall call the elements of Q operators. We write function division with a double dot, i.e., $f(t): g(t) \in Q$ if $f(t)$, $g(t) \in F$ and $g(t) \neq 0$, or we shall use a vinculum where function division is understood. F is isomorphically contained in Q so functions are operators. However, there are operators which are not functions, so F is not a field. For if F were a field then, given $f, h \in F$ with $f(0) = 0$, $h(0) \neq 0$ the equation $fx = h$ would have a solution $x \in F$. But $f(0) \cdot x(0) = h(0)$ yields a contradiction since $f(0) \cdot x(0) = 0 \cdot x(0) = 0$ and $h(0) \neq 0$.

Since Q is a field we have solutions for the equation

$$fx = h,$$

where $x \in Q$ and $f, h \in F$. Consider, in particular, the equation

$$tp = 1.$$

The solution, p, of this equation is not a function, but an operator and will play a fundamental role in what follows. Since t is the integration operator, we shall call its inverse operator, p, the differential operator.

<u>Theorem</u> 1.1. If f(t) is a continuous function and $f'(t) \varepsilon F$, then

$$pf(t) = f'(t) + pf(0).$$

<u>Proof</u>. We have

$$tf'(t) = \int_0^t f'(\tau)d\tau = f(t) - f(0),$$

and, since tp = 1, it follows that

$$pf(t) = f'(t) + pf(0).$$

Provided $f^{(n)}(t)$ exists, this result can be generalized to

$$p^n f(t) = f^{(n)}(t) + p^n f(0) + p^{n-1} f'(0) + \dots + pf^{(n-1)}(0). \qquad (1.5)$$

Theorem 1.1 can be shown to hold for merely continuous functions (1:116).

CHAPTER II

EXAMPLES

Before proceeding to some examples let us first establish basic relationships that we will need.

Since pt = 1, we can write the integration operator, t, in the form $\frac{1}{p}$, i.e.,

$$\frac{1}{p} f(t) = \int_0^t f(\tau)d\tau.$$

We obtain another operator if we take $f(t) = e^{\alpha t}$, where α is a constant. Then

$$\frac{1}{p} \cdot e^{\alpha t} = \int_0^t e^{\alpha\tau}d\tau = \frac{1}{\alpha}(e^{\alpha t}-1),$$

or

$$\frac{p}{p-\alpha} = e^{\alpha t}. \qquad (2.1)$$

It follows that

$$\frac{p}{(p-\alpha)^2} = \frac{1}{p} \cdot e^{\alpha t} \cdot e^{\alpha t}$$

$$= \int_0^t e^{\alpha\tau}e^{\alpha(t-\tau)}d\tau$$

$$\frac{p}{(p-\alpha)^2} = te^{\alpha t}. \qquad (2.2)$$

Now, if

$$\frac{p}{(p-\alpha)^{k+1}} = \frac{t^k}{k!} e^{\alpha t},$$

then

$$\frac{1}{p} \cdot \frac{p}{(p-\alpha)^{k+1}} \cdot \frac{p}{p-\alpha} = \frac{1}{k!} \int_0^t \tau^k e^{\alpha\tau} e^{\alpha(t-\tau)} d\tau$$

$$= \frac{e^{\alpha t}}{k!} \int_0^t \tau^k d\tau .$$

$$\frac{1}{p} \cdot \frac{p}{(p-\alpha)^{k+1}} \cdot \frac{p}{p-\alpha} = \frac{t^{k+1}}{(k+1)!} e^{\alpha t}.$$

Hence, we have established the relation

$$\frac{p}{(p-\alpha)^{n+1}} = \frac{t^n}{n!} e^{\alpha t}. \qquad (2.3)$$

Letting $f(t) = \sin\alpha t$ or $\cos \alpha t$ in (1.5) we get

$$p^2 \sin \alpha t = -\alpha^2 \sin \alpha t + p\alpha,$$

$$p^2 \cos \alpha t = -\alpha^2 \cos \alpha t + p^2$$

and

$$\frac{p\alpha}{p^2+\alpha^2} = \sin \alpha t, \quad \frac{p^2}{p^2+\alpha^2} = \cos \alpha t. \qquad (2.4)$$

Letting $\alpha = \beta + i\gamma$ in (2.1), we get

$$e^{\beta t+i\gamma t} = \frac{p}{p-\beta-i\gamma}$$

$$= \frac{p(p-\beta+i\gamma)}{(p-\beta)^2+\gamma^2} .$$

Since $e^{it} = \cos t + i \sin t$, separating into real and imaginary parts, we have

$$\frac{p(p-\beta)}{(p-\beta)^2+\gamma^2} = e^{\beta t} \cos \gamma t,$$

$$\frac{p\gamma}{(p-\beta)^2+\gamma^2} = e^{\beta t} \sin \gamma t. \qquad (2.5)$$

We are now in a position to consider some examples of the application of Mikusinski operators (2:1). The first example is of particular interest since the factor e^{t^2} appears on the right-hand side of the equation. Since e^{t^2} does not have a Laplace transform, we cannot use the Laplace transformation to solve this equation and, hence, the beauty of the Mikusinski operator is apparent.

Example 1. Solve the differential equation

$$y'(t) + 2y(t) = 2(t+1)e^{t^2}$$

with $y(0) = 1$.

First we write the equation in operator form using (1.5),

$$py - p + 2y = 2(t+1)e^{t^2}.$$

Solving for y, we get

$$y = \frac{p}{p+2} + \frac{2(t+1)e^{t^2}}{p+2} .$$

We can now write y as a function of t to obtain

$$\begin{aligned} y(t) &= e^{-2t} + 2\int_0^t e^{-2(t-\tau)}(\tau+1)e^{\tau^2}d\tau \\ &= e^{-2t} + 2e^{-2t-1}\int_0^t e^{(\tau+1)^2}(\tau+1)d\tau \\ &= e^{-2t} + e^{-2t-1+(\tau+1)^2}\Big|_0^t \\ &= e^{-2t} + e^{t^2} - e^{-2t} \\ &= e^{t^2}. \end{aligned}$$

We can easily see that this is the desired solution.

The following example reveals the use of the Mikusinski operators when dealing with systems of differential equations.

Example 2. Obtain the general solution to the following system of differential equations.

$$x'(t) + y(t) = t^2 + 6t + 1$$

$$y'(t) - x(t) = -3t^2 + 3t + 1$$

First we write the system in operator form, and assume $x(0) = x_0$, $y(0) = y_0$,

$$px + y = px_0 + \frac{2}{p^2} + \frac{6}{p} + 1$$

$$-x + py = py_0 - \frac{6}{p^2} + \frac{3}{p} + 1.$$

Now solve for y, then for x,

$$(p^2+1)y = (1+x_0)p + p^2y_0 + \frac{2}{p^2} + 4$$

$$y = (1+x_0)\frac{p}{p^2+1} + y_0\frac{p^2}{p^2+1} + \frac{2}{p^2} + \frac{2}{p^2+1}.$$

Using (2.4),

$$y(t) = (1+x_0)\sin t + y_0 \cos t + t^2 + 2(1-\cos t)$$

$$y(t) = (1+x_0)\sin t + (y_0-2)\cos t + t^2 + 2.$$

$$(p^2+1)x = p^2x_0 - py_0 + \frac{2}{p} + \frac{6}{p^2} + 6 - \frac{3}{p} + p - 1$$

$$x = x_0\frac{p^2}{p^2+1} + (1-y_0)\frac{p}{p^2+1} - \frac{1}{p(p^2+1)}$$

$$+ \frac{6}{p^2(p^2+1)} + \frac{5}{p^2+1}$$

$$x = x_0 \frac{p^2}{p^2+1} + (2-y_0)\frac{p}{p^2+1} + \frac{6}{p^2} - \frac{1}{p} - \frac{1}{p^2+1}$$

$$x(t) = (x_0+1)\cos t + (2-y_0)\sin t + 3t^2 - t - 1.$$

Hence, we have the solutions $x(t)$, $y(t)$ of our system of differential equations.

In the next two examples we consider some integral equations and find that the operators allow us to solve these equations quite easily.

<u>Example</u> 3. Obtain the solution to the integral equation

$$f(t) = \cos t - t - 2 - \int_0^t (t-\tau)f(\tau)d\tau.$$

First, rewrite the equation in operator form, yielding

$$f = \frac{p^2}{p^2+1} - \frac{1}{p} - 2 - \frac{f}{p^2}.$$

Solve for f, obtaining

$$f = \frac{p^4}{(p^2+1)^2} - \frac{p}{p^2+1} - \frac{2p^2}{p^2+1}$$

$$f(t) = (\cos t \cdot \cos t) - \sin t - 2\cos t$$

$$= \frac{d}{dt}\int_0^t \cos\tau\cos(t-\tau)d\tau - \sin t - 2\cos t$$

$$= \cos t - \frac{t}{2} \sin t - \sin t - 2 \cos t$$

$$= -\left[\cos t + \left(\frac{t}{2} + 1\right) \sin t\right].$$

<u>Example</u> 4. Find f, if it exists, in the integral equation

$$f(t) = \cos t + a \int_0^t \cos(t-\tau) f'(\tau) d\tau.$$

Since f(0) = 1, we rewrite the equation in the form

$$f = \frac{p^2}{p^2+1} + \frac{a}{p} \cdot \frac{p^2}{p^2+1} \cdot f'(t)$$

or

$$f = \frac{p^2}{p^2+1} + a \frac{p}{p^2+1} (pf-p).$$

Solving for f, we have

$$f = \frac{(1-a)p^2}{(1-a)p^2+1} ,$$

which can be put in the form

$$f = \frac{p^2}{p^2 + \left(\sqrt{\frac{1}{1-a}}\right)^2}$$

Hence by (2.4)

$$f(t) = \cos\left(\frac{t}{\sqrt{1-a}}\right), \quad a \neq 1.$$

Before considering a final example, some additional theory will be introduced. Although we considered only continuous functions with continuous derivatives in the development of the field Q, it can be shown that had we started with functions which are piecewise differentiable, having at most a finite number of points at which the derivative is either discontinuous or unbounded, the same results could have been accomplished, i.e., we still could have formed a field (1:114).

Let us now consider a field of functions of the above type with addition and multiplication defined as before.

One function that we encounter in this new set of functions is the displacement operator

$$v_\lambda(t) = \begin{cases} 0 \text{ if } t < \lambda \\ 1 \text{ if } t > \lambda \end{cases}$$

where $\lambda \geq 0$.

Notice that

$$v_\lambda g = \frac{d}{dt} \int_0^t v_\lambda(t-\tau) g(\tau) d\tau = 0$$

$$\text{for } t - \lambda < \tau, \text{ i.e., } t < \lambda$$

$$v_\lambda g = \frac{d}{dt} \int_0^{t-\lambda} g(\tau) d\tau = g(t-\lambda) \text{ for } t > \lambda.$$

Therefore,

$$v_\lambda g = \begin{cases} 0 \text{ for } t < \lambda \\ g(t-\lambda) \text{ for } t > \lambda. \end{cases} \tag{2.6}$$

In particular, for $g(t) = v_\beta(t)$ we have

$$v_\lambda v_\beta = v_{\lambda+\beta} \tag{2.7}$$

which holds for non-negative λ, β. When $\lambda = \beta$,

$$v_\lambda v_\lambda = v_\lambda^2 = v_{2\lambda}.$$

In general,

$$v_\lambda^n = v_{n\lambda},$$

for any positive integer n. If we put $\lambda = \beta/n$, we obtain

$$v_{\beta/n}^n = v_\beta,$$

or, extracting the n-th root

$$v_\beta^{1/n} = v_{\beta/n} .$$

If we put $\lambda = \beta/m$, where m is any positive integer, we get

$$v_{n\beta/m} = v_{\beta/m}^n = (v_\beta^{1/m})^n = v_\beta^{n/m},$$

i.e., $v_\beta^\alpha = v_{\alpha\beta}$ (2.8)

for any rational number $\beta > 0$. If we let α tend to an arbitrary positive real number we see that (2.8) is valid for positive irrational α, where we regard the expression on the left side as being defined by the expression on the right side. By (2.7), the usual laws for exponents are satisfied and, since $v_0 = 1$, we can apply (2.8) for $\alpha = 0$.

With the aid of the displacement operator, all piecewise linear functions can be expressed in terms of displacement and integration operators. Before considering an example, notice that we can write

$$g(t) = |t-a| = a - t + 2v_a t. \qquad (2.9)$$

Using (2.9) we can now write immediately

$$f(t) = t - |t-1|-2|t-2|$$

$$= t - \left[1-t+2v_1t\right] -2\left[2-t+2v_2t\right]$$

$$= 4t - 5 - 2v_1t - 4v_2t.$$

In operator form,

$$f = -5 + (4-2v_1-4v_2)\frac{1}{p} .$$

Periodic functions may be represented conveniently by means of the displacement operator. Let f(t) be a function of period s, i.e.,

$$f(t+s) = f(t).$$

The periodic function, f(t), is uniquely determined by the function

$$\phi(t) = \begin{cases} f(t) & \text{for } 0 \le t < s, \\ 0 & \text{for } s < t. \end{cases} \qquad (2.10)$$

In view of (2.9), f(t) and $\phi(t)$ are related by

$$\phi(t) + v_S f(t+s) = f(t)$$

or,

$$\phi(t) + v_S f(t) = f(t) \rightarrow f(t) = \frac{\phi(t)}{1-v_S} . \qquad (2.11)$$

Functions having the property

$$f(t+s) = -f(t)$$

with period 2s may be easily be expressed in terms of displacement operators, since

$$\phi(t) + v_s f(t+s) = f(t).$$

Hence,

$$\phi(t) + v_s\left[-f(t)\right] = f(t) \rightarrow f(t) = \frac{\phi(t)}{1+v_s}. \qquad (2.12)$$

We are now in a position to consider a final example.

Example 5. Let the function, f(t), be defined by

$$f(t) = (-1)^n(t-2n) \text{ for } 2n < t < 2n + 2,$$

where n = 0, 1, 2, ...

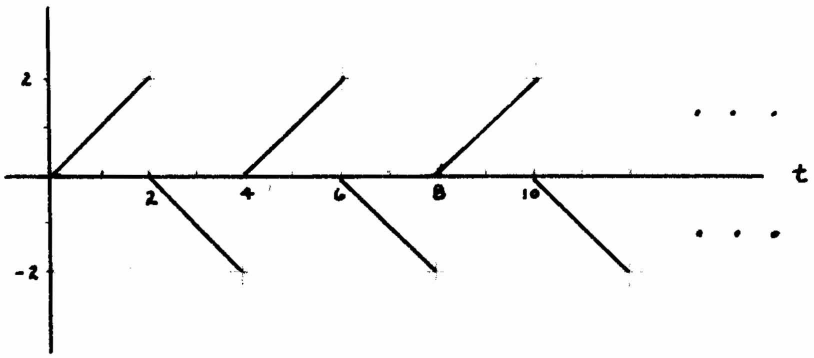
(a) Sketch f(t) and express it in operator form.

(b) Solve the differential equation

$$y'(t) + y(t) = f(t), \text{ with } y(0) = -1.$$

(c) Express the solution as a function of t and sketch its graph.

(a) Sketching f(t) we get



Since f(t) is a periodic function with period 4, by (2.11) we can write

$$f(t) = \frac{\phi(t)}{1-v_4},$$

where

$$\phi(t) = t + (-t-2)v_2 + (-t)v_2 + (t+2)v_4$$

$$= t - 2v_2(t+1) + (t+2)v_4.$$

In operator form, f becomes

$$f = \frac{1}{1-v_4}\left[\frac{1}{p} - \frac{2v_2}{p} - 2v_2 + \frac{v_4}{p} + 2v_4\right].$$

(b) To solve the differential equation we write

$$py - py(0) + y = f$$

or,

$$y = \frac{f}{p+1} - \frac{p}{p+1} .$$

Substituting for f, we get

$$y = \frac{1}{1-v_4} \cdot \frac{1}{p+1}\left[\frac{1}{p} - \frac{2v_2}{p} - 2v_2 + \frac{v_4}{p} + 2v_4\right] - \frac{p}{p+1}$$

$$= \frac{1}{1-v_4}\left[\frac{1}{p(p+1)} - \frac{2v_2}{p} - \frac{2v_2}{p+1} + \frac{v_4}{p(p+1)}\right.$$

$$\left. + \frac{2v_4}{p+1} - \frac{p(1-v_4)}{p+1}\right]$$

$$= \frac{1}{1-v_4}\left[\frac{1}{p} - \frac{1}{p+1} - \frac{2v_2}{p} + \frac{v_4}{p} + \frac{v_4}{p+1} - \frac{p}{p+1} + \frac{pv_4}{p+1}\right]$$

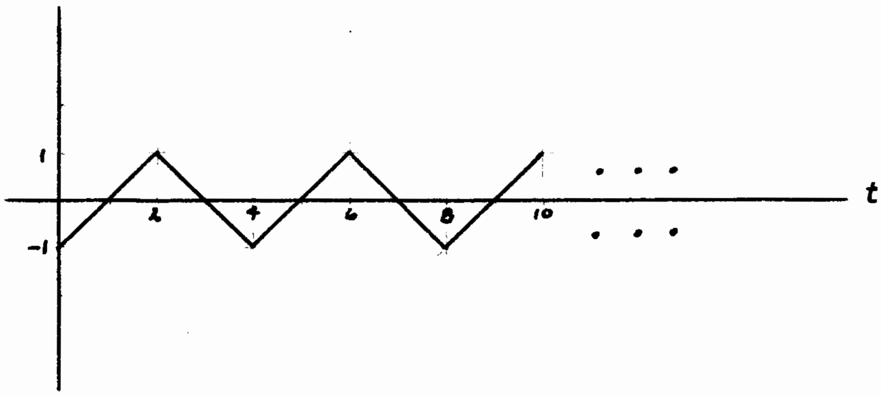
$$y(t) = \frac{1}{1-v_4}\left[t - (1-e^{-t}) - e^{-t} - 2v_2 t + v_4\left[t + (1-e^{-t})\right.\right.$$

$$\left.\left. + e^{-t}\right]\right]$$

$$= \frac{1}{1-v_4}\left[t - 1 - 2v_2 t + v_4(t+1)\right].$$

Since the factor $\frac{1}{1-v_4}$ appears we know that y(t) is a periodic function of period 4.

(c) Since y(t) is a periodic function we need only graph the function inside the braces.



Finally, write y as a function of t alone, obtaining

$$y(t) = (-1)^n(t-2n-1) \text{ for } 2n < t < 2n + 2.$$

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