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On the Theory of Quaternions

Earle David Smith
Central Washington University

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Earle David Smith

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APPROVED FOR THE GRADUATE FACULTY

Dale R. Comstock, COMMITTEE CHAIRMAN

Bernard L. Martin

R. F. Ruebel

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THE PROBLEM

Ordered triplets serve to describe space geometrically just as ordered pairs serve to describe the plane. A multiplication can be defined on pairs consistent with the complex numbers. However, as this paper will show, one cannot define a multiplication on triplets which is consistent with the complex numbers.

It was not until the early part of the Nineteenth Century that William Roland Hamilton created a system for describing space using ordered four-tuples (2:105). His theory employs the concept of three imaginaries, one of which corresponds to the imaginary i of the complex numbers. In fact, the structure of quaternions, Hamiltonian four-tuples, in the last analysis is seen to be an extension of the complex numbers. However, multiplication in the two systems does not in general have the same characteristics since quaternion multiplication is a non-commutative operation.

It was the purpose of this paper to investigate the structure of quaternions and examine their relationship to the real and complex numbers.

Attention was also given to the geometrical interpretation of quaternion multiplication and the amplitude of a quaternion. Applications related to vector analysis are also presented.

Although the author of this paper does not often refer to Hamilton by name, the material that follows is nevertheless an elaboration of his original work (2).

CHAPTER II

THE THEORY OF QUATERNIONS

The development of the theory of quaternions was greatly influenced by the properties characteristic of the complex numbers. The Law of Moduli is one such property which is stated as follows:

for any two complex numbers, a+bi and c+di, if $(a+bi)(c+di) = p+qi$, then $\sqrt{a^2+b^2}\sqrt{c^2+d^2} = \sqrt{p^2+q^2}$. Since quaternions are to be an extension of the complex numbers it is essential that quaternions satisfy the Law of Moduli.

In his early work with quaternions, Hamilton (1:103) was prompted to consider the existence of a second imaginary which, geometrically speaking, would be perpendicular to the complex plane at the origin. Denoting this second imaginary by the symbol j, space can then be resolved into a coordinate system involving the real-axis, i-axis and j-axis.

For any point in space with coordinates (a,b,c), the elements a, b, c are associated with the real-axis, i-axis, and j-axis, respectively. Further, just as the point (a,b) in the complex plane is associated with a+bi, let (a,b,c) in space be associated with a+bi+cj where we let

$$(a,b) = (a,b,0). \qquad (1)$$

Since $i^2=-1$, let $j^2=-1$ and consider the product $(a+bi+cj)(a+bi+cj)$. We have that

$(a+bi+cj)^2=a^2+abi+acj+abi-b^2-bcij+acj+bcji-c^2$. Therefore,

$$(a+bi+cj)^2= a^2-b^2-c^2+2abi+2acj+bcij+bcji \qquad (2)$$

using the fact that $i^2=j^2=-1$. Since $\sqrt{a^2+b^2+c^2}\ \sqrt{a^2+b^2+c^2} = \sqrt{(a^2-b^2-c^2)^2+(2ab)^2+(2ac)^2}$, equation (2) would satisfy the Law of Moduli if

i) the quantities $a^2-b^2-c^2$, $2ab$ and $2ac$ are associated with the real-axis, i-axis and j-axis, respectively; and

ii) $bcij+bcji=0$.

On the basis of i) and ii), equation (2) can be expressed as

$$(a+bi+cj)^2=a^2-b^2-c^2+2abi+2acj. \qquad (3)$$

Notice that whenever $c=0$, equation (3) becomes $(a+bi)^2= a^2-b^2+2abi$. From the assumption that ii) is true, it follows that either iii) $ij=ji=0$ or iv) $ij=-ji$. Since we want $i^2=j^2=-1$, we cannot adopt iii). Hence, $ij=-ji$.

Consider the more general product $(x+bi+cj)(y+bi+cj)$ where $i^2=j^2=-1$ and $ij=-ji$. We have that

$$\begin{aligned}(x+bi+cj)(y+bi+cj) &= xy+xbi+xcj+ybi-b^2+bcij+cyj+cbji-c^2\\ &= xy-b^2-c^2+b(x+y)i+c(x+y)j+bcij+bcji\\ &= xy-b^2-c^2+b(x+y)i+c(x+y)j.\end{aligned}$$

Since $\sqrt{x^2+b^2+c^2}\ \sqrt{y^2+b^2+c^2} = \sqrt{(xy-b^2-c^2)^2+(b(x+y))^2+(c(x+y))^2}$, the above equation is consistent with the Law of Moduli.

Finally, consider the product $(a+bi+cj)(w+xi+yj)$ where $i^2=j^2=-1$ and $ij =-ji$. We have that

$$(a+bi+cj)(w+xi+yj) = aw+axi+ayj+biw+bxi^2+byij+cwj+cxji+cyj^2$$
$$= aw-bx-cy+(ax+bw)i+(ay+cw)j+(by-cx)ij.$$

Thus,

$$(a+bi+cj)(w+xi+yj) = aw-bx-cy+(ax+bw)i+(ay+cw)j+(by-cx)ij. \quad (4)$$

After inspecting equation (4), one might suspect that $(by-cx)ij=0$, since there is not an ij-axis in our system. However, this would require that $ij=0$ which contradicts the fact $i^2=j^2=-1$. Furthermore, if $(by-cx)ij=0$, then equation (4) reduces to $(a+bi+cj)(w+xi+yj)=aw-bx-cy+(ax+bw)i+(ay+cw)j$ which satisfies the closure property but not the Law of Moduli. Thus, we must consider $(by-cx)ij\neq 0$ yielding $ij\neq 0$. If $ij\neq 0$, then i) $ij\neq j$, ii) $ij\neq i$ and iii) $ij\neq r$ for any real number r. These three conditions are true since $i^2=j^2=-1$. The symbol ij must therefore represent an imaginary different from either i or j. Let k denote a third imaginary such that $ij=k$ where $k^2=-1$ consistent with $i^2=j^2=-1$. Substituting k into equation (4) we obtain

$$(a+bi+cj)(w+xi+yj) = aw-bx-cy+(ax+bw)i$$
$$+(ay+cw)j+(by-cx)k. \quad (5)$$

Notice that the right side of equation (5) is of the form $p+qi+rj+sk$ whereas both factors on the left are of the form $l+mi+nj$. Hence, multiplication of triplets as defined by equation (5) is not a closed operation. To correct this discrepancy we need only express $a+bi+cj$ as $a+bi+cj+0k$ and

$w+xi+yj$ as $w+xi+yj+0k$ so that substitution into equation (5) yields

$$(a+bi+cj+0k)(w+xi+yj+0k) = aw-bx-cy+(ax+bw)i +(ay+cw)j+(by-cx)k. \qquad (6)$$

Notice that

$$\sqrt{a^2+b^2+c^2+o^2}\ \sqrt{w^2+x^2+y^2+o^2}=\sqrt{(aw-bx-cy)^2+(ax+bw)^2+(ay+cw)^2+(by-cx)^2}$$

which shows equation (6) is consistent with the Law of Moduli.

Thus far we have seen that the product of two triplets is not in general a triplet. This is shown by equation (5). Further, the closure property demands that multiplication of triplets of the form $a+bi+cj$ be thought of as multiplication of four-tuples of the form $a+bi+cj+dk$ where $d=0$. The product of two such four-tuples is given by equation (6). However, equation (6) fails to define the product of two four-tuples of the form $a+bi+cj+dk$ where $d\neq 0$. Therefore, consider the product of two such elements $a+bi+cj+dk$ and $w+xi+yj+zk$ where $i^2=j^2=k^2=-1$, $ij=-ji$ and $ij=k$.

For the product, we have

$$(a+bi+cj+dk)(w+xi+yj+zk) = aw+axi+ayj+azk+bwi+bxi^2 +byij+bzik+cwj+cxji+cyj^2 +czjk+dwk+dxki+dykj+dzk^2. \qquad (7)$$

Since $ij=k$ and $ik=i^2j$, it follows that $ik=-j$. In an analogous way to $ij=-ji$, we let $ik=-ki$. Then $j=ki$. If $j=ki$, then $kj=k^2i$ so that $-i=kj$. Again, as $ij=-ji$, let

$kj=-jk$ and we obtain $i=jk$. In summary

$$i^2=j^2=k^2=-1,$$
$$ij=-ji,\ ik=-ki,\ jk=-kj,$$
$$i=jk,\ j=ki \text{ and } k=ij. \tag{8}$$

These equations can equivalently be written as $i^2=j^2=k^2=ijk=-1$.

Using these identities, equation (7) can be expressed as

$$\begin{aligned}(a+bi+cj+dk)(w+xi+yj+zk) &= aw+axi+ayj+azk+bwi-bx \\ &\quad +byk-bzj+cwj-cxk-cy+czi \\ &\quad +dwk+dxj-dyi-dz \\ &= aw-bx-cy-dz+(ax+bw+cz-dy)i \\ &\quad +(ay-bz+cw+dx)j+(az+by-cx+dw)k.\end{aligned}$$

Hence, for any two four-tuples $a+bi+cj+dk$ and $w+xi+yj+zk$ where a, b, c, d, w, x, y, z are real numbers, we define $(a+bi+cj+dk)(w+xi+yj+zk) = p+qi+rj+sk$ where

$$p=aw-bx-cy-dz,\ r=ay-bz+cw+dx$$
$$q=ax+bw+cz-dy,\ s=az+by-cx+dw. \tag{9}$$

Of particular interest is the fact that equation (9) yields $(a,b,0,0)(w,x,0,0) = (aw-bx,\ ax+bw,\ 0,\ 0)$ which corresponds to $(a,b)(w,x) = (aw-bx,\ ax+bw)$ in the complex number system.

Therefore, let $a=(a,0,0,0)$, $(a,b)=(a,b,0,0)$, and $(a,b,c)=(a,b,c,0)$. Also, let $a(w,x,y,z)=(aw,ax,ay,az)$.

The last of these equations yields $-(w,x,y,z)=(-w,-x,-y,-z)$ with $a = -1$.

Since the system of quaternions is to be an extension of the complex numbers system, we want to extend complex addition to quaternion addition. Thus, for any two quaternions, $a+bi+cj+dk$ and $w+xi+yj+zk$, where a, b, c, d, w, x, y, z are real numbers, let

$$(a+bi+cj+dk)+(w+xi+yj+zk) = a+w+(b+x)i+(c+y)j+(d+z)k.$$

Subtraction of quaternions follows easily from addition since

$$\begin{aligned}(a+bi+cj+dk)-(w+xi+yj+zk)&=(a+bi+cj+dk)+(-(w+xi+yj+zk))\\ &=(a+bi+cj+dk)+(-w-xi-yj-zk)\\ &=a-w+(b-x)i+(c-y)j+(d-z)k.\end{aligned}$$

Division will be defined in Chapter III after the concept of multiplicative inverse is introduced. Now, using the definitions of multiplication and addition, the algebraic structure of quaternions will be examined.

CHAPTER III

THE ALGEBRAIC STRUCTURE OF QUATERNIONS

Let Q be the set given by $Q=\{(a,b,c,d)\,|\,a,b,c,d$ are real numbers$\}$. Q is called the set of quaternions. On Q, define =, +, - and $\cdot$ by

$(a,b,c,d)=(x,y,z,w)$ iff $a=x$, $b=y$, $c=z$, $d=w$;

$(a,b,c,d)+(x,y,z,w)=(a+x,b+y,c+z,d+w)$; and

$(a,b,c,d)(x,y,z,w)=(p,q,r,s)$

where p,q,r,s are given by equation (9). The operations of multiplication and addition are clearly binary operations. Consider now the system (Q,+) consisting of the set Q together with the binary operation + defined on Q.

For any (a,b,c,d) Q we have $(a,b,c,d)+(0,0,0,0)=(0,0,0,0)+(a,b,c,d)=(a,b,c,d)$. Consequently, (0,0,0,0) is the identity element for addition.

If $(a,b,c,d)+(w,x,y,z)=(0,0,0,0)$, then $w=-a$, $x=-b$, $y=-c$ and $z=-d$. Thus, $(-a,-b,-c,-d)$ is the additive inverse of (a,b,c,d). $(-a,-b,-c,-d)$ will be denoted by $-(a,b,c,d)$.

Addition and multiplication are commutative and associative in the reals. Hence, quaternion addition will have these properties. As a result of these four conditions, we may characterize (Q,+) as an Abelian group.

Now consider the system $(Q,+,\cdot)$ consisting of the set Q and the binary operations + and $\cdot$ defined on Q. Let

$p=(a,b,c,d)$, $q=(e,f,g,h)$ and $r=(x,y,z,w)$. Then, we have

$$
\begin{aligned}
p(qr)&=(a,b,c,d)((e,f,g,h)(x,y,z,w))\\
&=(a,b,c,d)(ex-fy-gz-hw,ey+fx+gw-hz,ez-fw+gx+hy,\\
&\quad ew+fz-gy+hx)\\
&=(a(ex-fy-gz-hw)-b(ey+fx+gw-hz)-c(ez-fw+gx+hy)\\
&\quad -d(ew+fz-gy+hx),\\
&\quad a(ey+fx+gw-hz)+b(ex-fy-gz-hw)+c(ew+fz-gy+hx)\\
&\quad -d(ez-fw+gx+hy),\\
&\quad a(ez-fw+gx+hy)-b(ew+fz-gy+hx)+c(ex-fy-gz-hw)\\
&\quad +d(ey+fx+gw-hz),\\
&\quad a(ew+fz-gy+hx)+b(ez-fw+gx+hy)-c(ey+fx+gw-hz)\\
&\quad +d(ex-fy-gz-hw))\\
&=(aex-afy-agz-ahw-bey-bfx-bgw+bhz-cez+cfw-cgx-chy-dew\\
&\quad -dfz+dgy-dhx,\\
&\quad aey+afx+agw-ahz+bez-bfy-bgz-bhw+cew+cfz-cgy+chx-dez\\
&\quad +dfw-dgx-dhy,\\
&\quad aez-afw+agx+ahy-bew-bfz+bgy-bhx+cex-cfy-cgz-chw+dey\\
&\quad +dfx+dgw-dhz,\\
&\quad aew+afz-agy+ahx+bez-bfw+bgx+bhy-cey-cfx-cgw+chz+dex\\
&\quad -dfy-dgz-dhw).
\end{aligned}
$$

Also,

$$
\begin{aligned}
(pq)r&=((a,b,c,d)(e,f,g,h))(x,y,z,w)\\
&=(ae-bf-cg-dh,af+be+ch-dg,ag-bh+ce+df,ah+bg-cf+de)(x,y,z,w)\\
&=((ae-bf-cg-dh)x-(af+be+ch-dg)y-(ag-bh+ce+df)z\\
&\quad -(ah+bg-cf+de)w,
\end{aligned}
$$

$(ae-bf-cg-dh)y+(af+be+ch-dg)x+(ag-bh+ce+df)w$
$-(ah+bg-cf+de)z,$
$(ae-bf-cg-dh)z-(af+be+ch-dg)w+(ag-bh+ce+df)x$
$+(ah+bg-cf+de)y,$
$(ae-bf-cy-dh)w+(af+be+ch-dg)z-(ag-bh+ce+df)y$
$+(ah+bg-cf+de)x)$
$=(aex-bfx-cgx-dhx-afy-bey-chy+dgy-agz+bhz-cez-dfz$
$-ahw-bgw+cfw-dew,$
$aey-bfy-cgy-dhy+afx+bex+chx-dgx+agw-bhw+cew+dfw$
$-ahz-bgz+cfz-dez,$
$aez-bfz-cgz-dhz-afw-bew-chw+dgw+agx-bhx+cex+dfx$
$+ahy+bgy-cfy+dey,$
$aew-bfw-cgw-dhw+afz+bez+chz-dgz-agy+bhy-cey-dfy$
$+ahx+bgx-cfx+dex).$

Since $p(qr)=(pq)r$, quaternion multiplication is associative. To establish that $(Q,+,\cdot)$ is a ring, it must be also shown that $p(q+r)=pq+pr$. We have

$p(q+r)=(a,b,c,d)((e,f,g,h)+(x,y,z,w))$
$=(a,b,c,d)(e+x,f+y,g+z,h+w)$
$=(a(e+x)-b(f+y)-c(g+z)-d(n+w),$
$a(f+y)+b(e+x)+c(n+w)-d(g+z)$
$a(g+z)-b(n+w)+c(e+x)+d(f+y),$
$a(h+w)+b(g+z)-c(f+y)+d(e+x))$
$=(ae+ax-bf-by-cg-cz-dh-dw,$
$af+ay+be+bx+ch+cw-dg-dz,$

$ag+az-bh-bw+ce+cx+df+dy,$

$ah+aw+bg+bz-cf-cy+de+dx)$

and

$$\begin{aligned}pq+pr&=(a,b,c,d)(x,y,z,w)+(a,b,c,d)(e,f,g,h)\\&=(ax-by-cz-dw,ay+bx+cw-dz,az-bw+cx+dy,aw+bz-cy+dx)\\&\quad+(ae-bf-cg-dh,af+be+ch-dg,ag-bh+ce+df,ah+bg-cf+de)\\&=(ax-by-cz-dw+ae-bf-cg-dh,\\&\quad ay+bx+cw-dz+af+be+ch-dg,\\&\quad az-bw+cx+dy+ag-bh+ce+df,\\&\quad aw+bz-cy+dx+ah+bg-cf+de).\end{aligned}$$

Thus $p(q+r)=pq+pr$ and we have established that $(Q,+,\cdot)$ is a ring.

Since $(a,b,c,d)(x,y,z,w)=(ax-by-cz-dw,ay+bx+cw-dz,$
$az-bw+cx+dy,aw+bz-cy+dx)$

and $(x,y,z,w)(a,b,c,d)=(xa-by-cz-dw,xb+ya+dz-wc,$
$xc-yd+za+wb,xd+yc-zb+wa),$

it follows that $(Q,+,\cdot)$ is a non-commutative ring.

Let us investigate the existence of an element (x,y,z,w) such that for any $(a,b,c,d)\varepsilon Q$, $(a,b,c,d)(x,y,z,w)=(x,y,z,w)(a,b,c,d)=(a,b,c,d)$. This requires that

$$\begin{aligned}&ax-by-cz-dw=a=xa-by-cz-dw\\&ay+bx+cw-dz=b=xb+ya+dz-wc\\&az-bw+cx+dy=c=xc-yd+za+wb\\&aw+bz-cy+dx=d=xd+ya-zb+wa\end{aligned}$$

which reduce to

$$2dz-2wc=0$$

$$2bw-2yd=0$$

$$2bz-2cy=0.$$

Since $(x,y,0,0)$ corresponds to (x,y), and the identity for multiplication in the complex number system is $(1,0)$, we must require the identity (x,y,z,w) to be of the form $(1,0,z,w)$. Taking $x=1$, $y=0$ the above equations reduce to

$$2dz-2wc=0$$

$$2bw=0$$

$$2bz=0$$

for all values of b, c, d. From the last two equations we have $w=z=0$. Thus, if $x=1$ and $y=z=w=0$, then the identity element for multiplication is $(1,0,0,0)$ and

$(a,b,c,d)(1,0,0,0)=(1,0,0,0)(a,b,c,d)=(a,b,c,d)$.

If $(a,b,c,d)\neq(0,0,0,0)$, then an element (x,y,z,w) is called the multiplicative inverse of (a,b,c,d) provided $(a,b,c,d)(x,y,z,w)=(x,y,z,w)(a,b,c,d)=(1,0,0,0)$. This would require that

$$\begin{aligned} ax-by-cz-dw&=1 \\ ay-bx+cw-dz&=0 \\ az-bw+cx+dy&=0 \\ aw+bz-cy+dx&=0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} ax-by-cz-dw&=1 \\ bx+ay-dz+cw&=0 \\ cx+dy+az-bw&=0 \\ dx-cy+bz+aw&=0. \end{aligned}$$

From the second system of equations, multiplication of the first and second rows by c and d, respectively, yields upon addition

$$
\begin{aligned}
(ac+bd)x-(bc-ad)y-(c^2+d^2)z&=c\\
bx \quad +ay \quad -dz \quad +cw &=0\\
cx \quad +dy \quad +az \quad -bw &=0\\
dx \quad -cy \quad +bz \quad +aw &=0.
\end{aligned}
$$

Similarly, multiplication of the third and fourth rows by a and b, respectively, yields upon addition

$$
\begin{aligned}
(ac+bd)x-(bc-ad)y-(c^2+d^2)z &= c\\
bx \quad +ay \quad -dz \quad +cw &= 0\\
cx \quad +dy \quad +az \quad -bw &= 0\\
(ac+bd)x+(ad-bc)y+(a^2+b^2)z &= 0.
\end{aligned}
$$

Subtracting the first row from the fourth, we have $(a^2+b^2)z+(c^2+d^2)z=-c$ and $z = \frac{-c}{a^2+b^2+c^2+d^2}$ provided that $a^2+b^2+c^2+d^2\neq 0$. Transforming the original system to

$$
\begin{aligned}
(a^2+b^2)x-(ac+bd)z-(ad-bc)w &= a\\
bx \quad +ay \quad +dz \quad +cw &= 0\\
(c^2+d^2)x+(ac+db)z +(ad-bc)w&=0\\
dx \quad -cy \quad +bz \quad +aw &= 0
\end{aligned}
$$

we have that $(a^2+b^2)x+(c^2+d^2)x = a$. Hence, $x=\frac{a}{a^2+b^2+c^2+d^2}$, provided that $a^2+b^2+c^2+d^2\neq 0$. Similar transformations of the original system of equations will yield

$y = \frac{-b}{a^2+b^2+c^2+d^2}$ and $w = \frac{-d}{a^2+b^2+c^2+d^2}$. Hence, if $(a,b,c,d)\neq(0,0,0,0)$, then its inverse element with respect to multiplication is given by $\frac{a}{u^2},\frac{-b}{u^2},\frac{-c}{u^2},\frac{-d}{u^2}$ where $u^2=a^2+b^2+c^2+d^2$. Notice that

$(\frac{a}{u^2}, \frac{-b}{u^2}, \frac{-c}{u^2}, \frac{-d}{u^2}) = \frac{1}{u^2}(a,-b,-c,-d)$.

However, for simplicity the element $(a,b,c,d)^{-1}$ will also be used to denote the multiplicative inverse of (a,b,c,d) when it exists.

The properties so far considered show that $(Q,+,\cdot)$ is a skew or non-commutative field. The field has no zero divisors for if

$$(a,b,c,d)(x,y,z,w)=(0,0,0,0) \text{ and }$$

$$(a,b,c,d)\neq(0,0,0,0), \text{ then}$$

$$(a,b,c,d)^{-1}(a,b,c,d)(x,y,z,w)=(a,b,c,d)^{-1}(0,0,0,0)$$

so that

$$(x,y,z,w)=(0,0,0,0).$$

Having now characterized the system $(Q,+,\cdot)$ as a skew field, we observe further that $(Q,+,\cdot)$ can be thought of as a four-dimensional vector space over the reals. Since this vector space has a bilinear and associative multiplication $(\cdot)$ with unity and an inverse for each non-zero element, it represents a division algebra. Quaternions are important in algebra since they are the only non-commutative division algebra over the reals (1:225).

With the algebraic structure of quaternions now established, the attention of this paper will be directed towards the examination of the geometrical interpretation of quaternions.

CHAPTER IV

THE GEOMETRICAL INTERPRETATION OF QUATERNIONS

In order to understand the geometrical representation of quaternions one might first examine the relationships found in the complex number system.

Interpret $i=(0,1)$ and $1=(1,0)$ as unit vectors in the complex plane and consider the various products involving $\pm i$ and ± 1. Whenever $\pm i$ is a factor, the product is obtained by a ninety degree rotation from the other factor in the product. Further, $(\pm i)^2$ is coincidental with -1.

In $(Q,+,\cdot)$ where $(a,b,c,d)=a+bi+cj+dk$, any quaternion can be thought of as a linear combination of the unit vectors 1, i, j and k where $1=(1,0,0,0)$, $i=(0,1,0,0)$, $j=(0,0,1,0)$ and $k=(0,0,0,1)$. Let these unit vectors be inclined ninety degrees to one another with -1, $-i$, $-j$ and $-k$ being their opposite vectors.

Consider now a coordinate system consisting of four perpendicular axes denoted by the real-axis, i-axis, j-axis and k-axis such that each axis is wholly determined by its corresponding positive and negative unit vectors. For this coordinate system, consider the identities:

i) $i^2=j^2=k^2=-1$

ii) $ij=-ji$, $ik=-ki$, $jk=-kj$

iii) $i=jk$, $j=ki$, $k=ij$.

These statements may be geometrically interpreted as follows:

i) any unit vector acting on itself experiences a rotation of ninety degrees onto the negative of the unit vector;

ii) if two factors of a product are interchanged, then the direction of the resulting product is opposite the original product; and

iii) the product of two different imaginaries is perpendicular to the plane containing its factors. The product is obtained by rotating the second factor ninety degrees around the first factor in the same direction, right or left, that j must be rotated ninety degrees around i to be coincidental with k. Statement iii) is refered to as the Rule of Rotation. The expression ki=j implies that a ninety degree rotation of i around the k-axis will make i and j coincidental. The direction of rotation is determined by the Rule of Rotation. With the use of this Rule and a diagram, one can easily see why condition ii) is true. Since each unit vector is perpendicular to the other three, one could select any three to form a three dimensional coordinate system. Consider then a coordinate system composed of the i, j and k axes satisfying iii).

In a three dimensional coordinate system having axes x, y, z, one normally constructs a spherical coordinate system as follows. Let z denote the vertical axis and y the horizontal axis such that for any point (r,s,t) the components r, s, t are associated with the x, y, z axes, respectively. For any point (r,s,t), let $m=\sqrt{r^2+s^2+t^2}$. If ℓ is the segment determined by (r,s,t) and the origin, then let α be the angle between ℓ and the z-axis and β be the angle between the projection of ℓ on the xy-plane and the x-axis. These conventions yield the identities

$$r=m \sin \alpha \cos \beta,$$

$$s=m \sin \alpha \sin \beta, \text{ and}$$

$$t=m \cos \alpha. \qquad (10)$$

Consider now the problem of setting up a spherical coordinate system in a system having four perpendicular axes denoted by 1, i, j and k.

Given a point (a,b,c,d) in the 1,i,j,k coordinate system, let real(a,b,c,d)=a, and imag(a,b,c,d)=(0,b,c,d). Thus, imag(a,b,c,d) is similar to (r,s,t) above in the sense that both are points that can be represented in three space. Let f denote the "segment" determined by the point (a,b,c,d) and (0,0,0,0). The "angle" between "segment" f and the positive real axis will be represented by θ. Let ℓ denote the segment determined by imag(a,b,c,d) and (0,0,0,0) such that the length of ℓ is given by $u \sin \theta$ where $u=\sqrt{a^2+b^2+c^2+d^2}$. Also, let $a=u \cos \theta$. Notice that whenever

$\theta=\pi/2$, the point (a,b,c,d) is ninety degrees removed from the real-axis. This means (a,b,c,d) is contained in one of the imaginary axes. Hence, (a,b,c,d) must be of the form (0,b,c,d) which is consistent with the fact $a=u \cos \theta$. Further, if $\theta=\pi/2$, then $u=\sqrt{b^2+c^2+d^2}$, and $\sin \theta = 1$. Thus, the length of ℓ as given by $u \sin \theta$ is consistent with the length of ℓ as given by the distance formula in three space.

Consider now the coordinate system determined by the i, j and k axes. Let i represent the vertical axis and k the horizontal axis. Further, let ϕ denote the angle between the segment ℓ, as described above, and the positive i-axis and let ψ denote the angle between the projection of segment ℓ on the jk-plane and the positive k-axis. Notice that $u \sin \theta$, ϕ and ψ in the i, j, k system correspond to m, α and β used in the x, y, z system. Just as equations (10) represent the spherical coordinates for the point (r,s,t), the following equations represent the spherical coordinates for the point (a,b,c,d) in the 1,i,j,k system:

$$\begin{aligned} a &= \cos \theta \\ b &= u \sin \theta \cos \phi \\ c &= u \sin \theta \sin \phi \cos \psi \\ d &= u \sin \theta \sin \phi \sin \psi. \end{aligned} \tag{11}$$

The quantities θ, u, ϕ and ψ are called the amplitude, modulus, colatitude and longitude of the quaternion (a,b,c,d).

Equations (11) are consistent with spherical coordinates in three space. For if $\theta=\pi/2$, then (a,b,c,d) becomes (0,b,c,d) which is a point that can be represented using only the i, j and k axes. Hence, (0,b,c,d) in the 1,i,j,k system corresponds to (b,c,d) in the i,j,k system. Setting $\theta=\pi/2$ reduce equations (11) to

$$a = 0,$$
$$b = \sqrt{b^2+c^2+d^2}\cos\phi,$$
$$c = \sqrt{b^2+c^2+d^2}\sin\phi\cos\psi,$$
$$d = \sqrt{b^2+c^2+d^2}\sin\phi\sin\psi,$$

which give the spherical coordinates for (b,c,d) in the i,j,k system. Using equations (11) it is possible to compute the angle between two vectors determined by the points (0,b,c,d) and (0,y,z,w).

Let (0,b,c,d) and (0,y,z,w) determine two vectors whose origin is (0,0,0,0). Since all three points have first component zero, we can consider instead the points (b,c,d), (y,z,w) and (0,0,0) in the i,j,k system as shown in Figure 1. Let A=(b,c,d), B=(0,0,0) and C=(y,z,w) where angle ABC has a measure of α.

The Law of Cosines requires that

$$\cos\alpha = \frac{\overline{AB}^2 + \overline{BC}^2 - \overline{AC}^2}{2\,\overline{AB}\;\overline{BC}}$$

where $\overline{AB}$ is the length of the segment whose endpoints are A and B. Using the distance formula we have that

$$\overline{AC}^2=(b-y)^2+(c-z)^2+(d-w)^2.$$

From equations (11) we obtain

$$b = u \sin\theta \cos\phi \qquad y = \bar{u} \sin\bar{\theta} \cos\bar{\phi}$$

$$c = u \sin\theta \sin\phi \cos\psi \qquad z = \bar{u} \sin\bar{\theta} \sin\bar{\phi} \cos\psi$$

$$d = u \sin\theta \sin\phi \sin\psi \qquad w = \bar{u} \sin\bar{\theta} \sin\bar{\phi} \sin\psi$$

where u, θ, ϕ, ψ and $\bar{u}$, $\bar{\theta}$, $\bar{\phi}$, $\bar{\psi}$ represent the modulus, amplitude, colatitude and longitude of $(0,b,c,d)$ and $(0,y,z,w)$, respectively. Hence,

$$\begin{aligned}
\overline{AC}^2=&(u \sin\theta \cos\phi - \bar{u} \sin\bar{\theta} \cos\bar{\phi})^2\\
&+(u \sin\theta \sin\phi \cos\psi - \bar{u} \sin\bar{\theta} \sin\bar{\phi} \cos\bar{\psi})^2\\
&+(u \sin\theta \sin\phi \sin\psi - \bar{u} \sin\bar{\theta} \sin\bar{\phi} \sin\bar{\psi})^2\\
=&u^2 \sin^2\theta \cos^2\phi + \bar{u}^2 \sin^2\bar{\theta} \cos^2\bar{\phi} -2u\bar{u} \sin\theta \sin\bar{\theta} \cos\phi \cos\bar{\phi}\\
&+u^2 \sin^2\theta \sin^2\phi \cos^2\psi + \bar{u}^2 \sin^2\bar{\theta} \sin^2\bar{\phi} \cos^2\bar{\psi}\\
&-2u\bar{u} \sin\theta \sin\bar{\theta} \sin\phi \sin\bar{\phi} \cos\psi \cos\bar{\psi}\\
&+u^2 \sin^2\theta \sin^2\phi \sin^2\psi + \bar{u}^2 \sin^2\bar{\theta} \sin^2\bar{\phi} \sin^2\bar{\psi}\\
&-2u\bar{u} \sin\theta \sin\bar{\theta} \sin\phi \sin\bar{\phi} \sin\psi \sin\bar{\psi}\\
=&u^2 \sin^2\theta(\cos^2\phi + \sin^2\phi \cos^2\psi + \sin^2\phi \sin^2\psi)\\
&+\bar{u}^2 \sin^2\bar{\theta}(\cos^2\bar{\phi} + \sin\bar{\phi} \cos^2\bar{\psi} + \sin^2\bar{\phi} \sin^2\bar{\psi})\\
&-2u\bar{u} \sin\theta \sin\bar{\theta}(\cos\phi \cos\bar{\phi} + \sin\phi \sin\bar{\phi} \cos\psi \cos\bar{\psi}\\
&+ \sin\phi \sin\bar{\phi} \sin\psi \sin\bar{\psi})\\
=&u^2 \sin^2\theta + \bar{u}^2 \sin^2\bar{\theta}\\
&-2u\bar{u} \sin \sin\bar{\theta}(\cos\phi \cos\bar{\phi} + \sin\phi \sin\bar{\phi} \cos\psi \cos\bar{\psi}\\
&+\sin\phi \sin\bar{\phi} \sin\psi \sin\bar{\psi})
\end{aligned}$$

since $\cos^2\phi + \sin^2\phi \cos^2\psi + \sin^2\phi \sin^2\psi = 1$. Note also

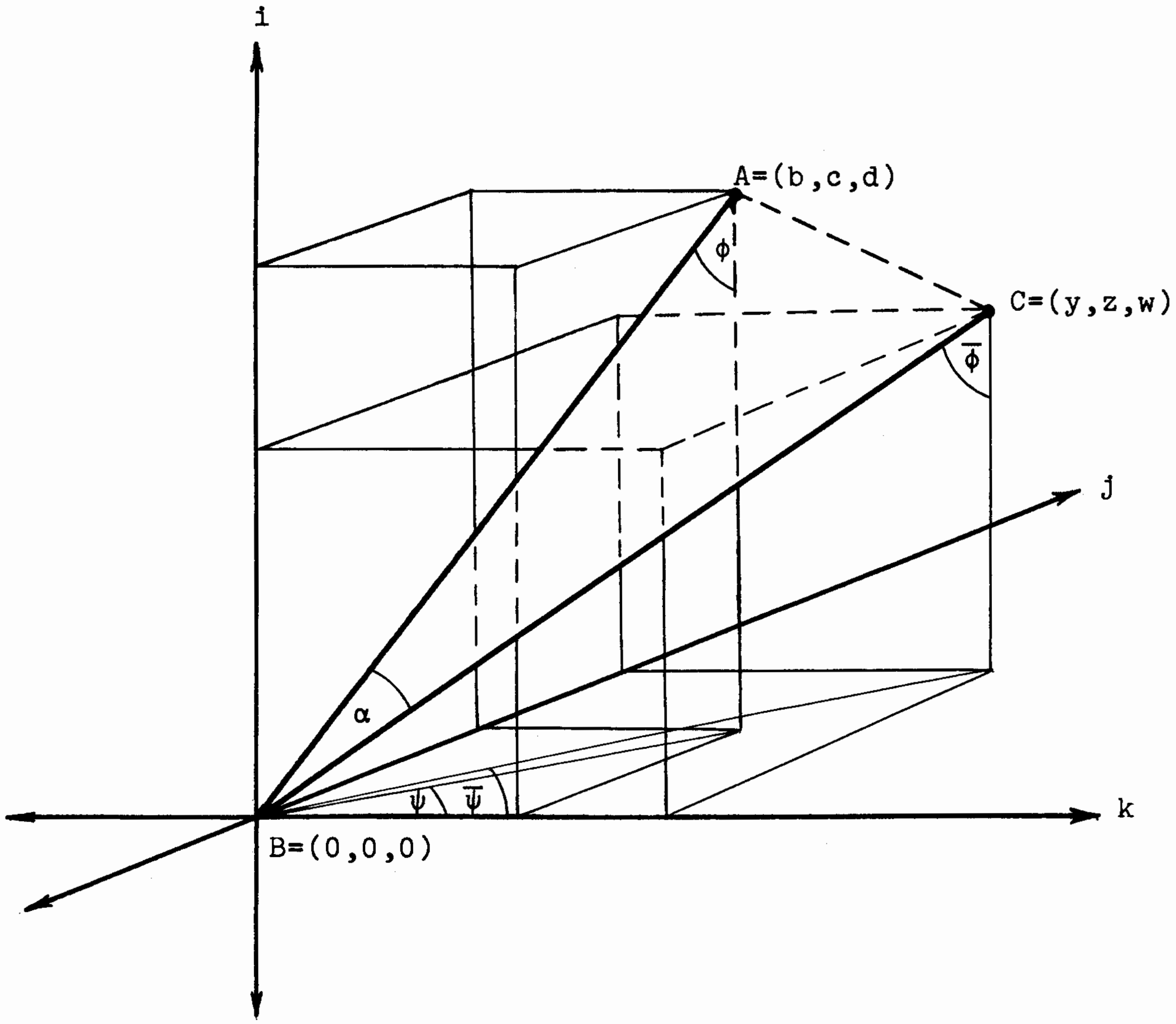

FIGURE 1

DETERMINING THE ANGLE α BETWEEN
TWO QUATERNIONS OF THE FORM
(0,b,c,d) AND (0,y,z,w)

that

$$\cos\theta\cos\bar{\theta} + \sin\phi\sin\bar{\Phi}\cos\psi\cos\bar{\Psi} + \sin\phi\sin\bar{\Phi}\sin\psi\sin\bar{\Psi}$$

can be reduced to

$\cos\theta\cos\bar{\theta} + \sin\theta\sin\bar{\theta}(\cos\psi\cos\bar{\Psi} + \sin\psi\sin\bar{\Psi})$ or

$\cos\theta\cos\bar{\theta} + \sin\theta\sin\bar{\theta}\cos(\psi - \bar{\Psi})$ so that substitution

into the last equation for $\overline{AC}^2$ yields

$$\overline{AC}^2 = u^2\sin^2\theta + \bar{u}^2\sin\bar{\theta}$$
$$-2u\bar{u}\sin\theta\sin\bar{\theta}(\cos\theta\cos\bar{\theta} + \sin\theta\sin\bar{\theta}\cos(\psi - \bar{\Psi})).$$

By definition, $\overline{AB} = u\sin\theta$ and $\overline{BC} = u\sin\theta$ so we have that

$$\overline{AB}^2 + \overline{BC}^2 - \overline{AC}^2 = 2u\bar{u}\sin\theta\sin\bar{\theta}(\cos\phi\cos\bar{\Phi}$$
$$+ \sin\phi\sin\bar{\Phi}\cos(\psi - \bar{\Psi})),$$ and

$2\,\overline{AB}\,\overline{BC} = 2u\bar{u}\sin\theta\sin\bar{\theta}$. Substituting these last two equations into the expression for $\cos\alpha$, we obtain

$$\cos\alpha = \cos\phi\cos\bar{\Phi} + \sin\phi\sin\bar{\Phi}\cos(\psi - \bar{\Psi}). \tag{12}$$

The angle between two vectors $bi+cj+dk$ and $yi+zj+wk$ can also be expressed in terms of $b,c,d,y,z,$ and w.

From Figure 1, page 22, we see that

$$\cos\phi = \frac{b}{\sqrt{b^2+c^2+d^2}}, \qquad \cos\bar{\Phi} = \frac{y}{\sqrt{y^2+z^2+w^2}},$$

$$\sin\phi = \frac{\sqrt{d^2+c^2}}{\sqrt{b^2+c^2+d^2}}, \qquad \sin\bar{\Phi} = \frac{\sqrt{z^2+w^2}}{\sqrt{y^2+z^2+w^2}},$$

$$\cos\psi = \frac{d}{\sqrt{c^2+d^2}}, \qquad \cos\bar{\Psi} = \frac{w}{\sqrt{z^2+w^2}},$$

$$\sin\psi = \frac{c}{\sqrt{c^2+d^2}}, \qquad \sin\bar{\Psi} = \frac{z}{\sqrt{z^2+w^2}},$$

which when substituted into equation (12) yield

$$\cos\alpha = \frac{by}{\sqrt{b^2+c^2+d^2}\ \sqrt{y^2+z^2+w^2}} + \frac{\sqrt{c^2+d^2}\ \sqrt{z^2+w^2}}{\sqrt{b^2+c^2+d^2}\ \sqrt{y^2+z^2+w^2}} \cdot \frac{cz+dw}{\sqrt{c^2+d^2}\ \sqrt{z^2+w^2}}$$

since $\cos(\psi - \overline{\psi}) = \cos\psi\cos\overline{\psi} + \sin\psi\sin\overline{\psi}$

$$= \frac{cz+dw}{\sqrt{c^2+d^2}\ \sqrt{z^2+w^2}}\ .$$

Therefore,

$$\cos\alpha = \frac{by+cz+dw}{\sqrt{b^2+c^2+d^2}\ \sqrt{y^2+z^2+w^2}}\ . \tag{13}$$

Equations (12) and (13) can now be used to interpret the geometrical representation of the amplitude of a quaternion three space.

Consider three quaternions (a,b,c,d), (x,y,z,w) and (p,q,r,s) such that (a,b,c,d)(x,y,z,w)=(p,q,r,s) where p, q, r, and s are given by equation (9). We have that imag(a,b,c,d)=(0,b,c,d), imag(x,y,z,w)=(0,y,z,w) and imag(p,q,r,s)=(0,q,r,s). If we use the i,j,k coordinate system, these quantities can then be represented by (b,c,d), (y,z,w) and (q,r,s), respectively. Using Figure 2, page 25, we see that (b,c,d), (y,z,w) and (q,r,s) determine vectors whose common origin is (0,0,0). About (0,0,0) visualize a sphere with radius one and center (0,0,0). Let A, B, and C represent the intersections of the vectors determined by (b,c,d), (y,z,w) and (q,r,s) with the sphere. Note that

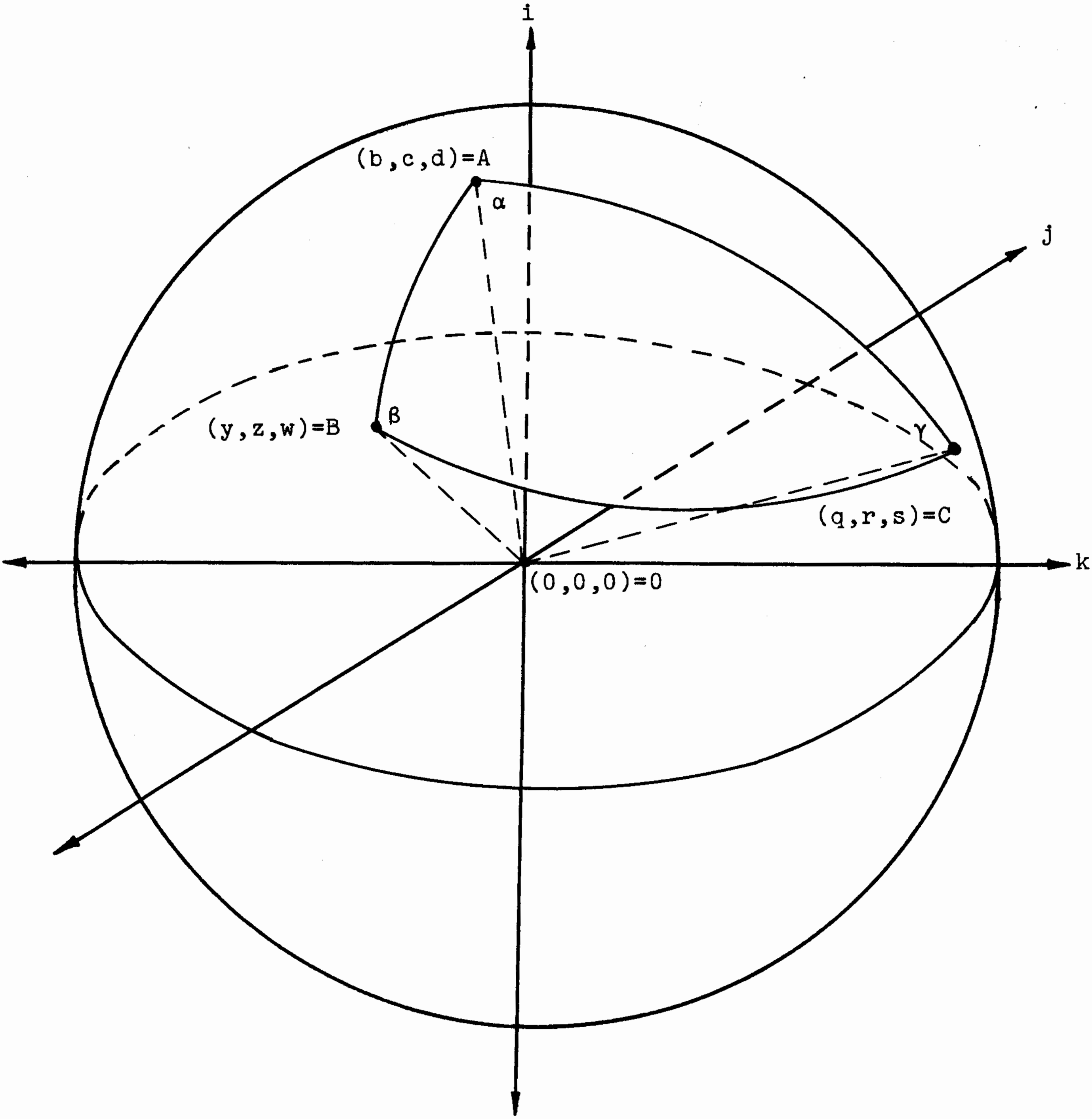

FIGURE 2

THE AMPLITUDE OF QUATERNIONS
RELATED TO THE ANGLES OF
A SPHERICAL TRIANGLE

some vectors may have to be extended to yield an intersection. In any case, points A, B and C determine a spherical triangle.

From equation (9) we have that

$$p = ax - by - cz - dw$$

$$q = ay + bx + cw - dz$$

$$r = az - bw + cx + dy$$

$$s = aw + bz - cy + dx.$$

By equations (11),

$$a = u\cos\theta \qquad x = \bar{u}\cos\bar{\theta}$$

$$b = u\sin\theta\cos\phi \qquad y = \bar{u}\sin\theta\cos\bar{\phi}$$

$$c = u\sin\theta\sin\phi\cos\psi \qquad z = \bar{u}\sin\bar{\theta}\sin\bar{\phi}\cos\bar{\psi}$$

$$d = u\sin\theta\sin\phi\sin\psi, \qquad w = \bar{u}\sin\bar{\theta}\sin\bar{\phi}\sin\bar{\psi},$$

$$p = \bar{\bar{u}}\cos\bar{\bar{\theta}}$$

$$q = \bar{\bar{u}}\sin\bar{\bar{\theta}}\cos\bar{\bar{\phi}}$$

$$r = \bar{\bar{u}}\sin\bar{\bar{\theta}}\sin\bar{\bar{\phi}}\cos\bar{\bar{\psi}}$$

$$s = \bar{\bar{u}}\sin\bar{\bar{\theta}}\sin\bar{\bar{\phi}}\sin\bar{\bar{\psi}}.$$

Combining these equations we have that

$$\begin{aligned}
\bar{\bar{u}}\cos\bar{\bar{\theta}} &= u\cos\theta\,\bar{u}\cos\bar{\theta} - u\sin\theta\cos\phi\,\bar{u}\sin\bar{\theta}\cos\bar{\phi} \\
&\quad -u\sin\theta\sin\phi\cos\psi\,\bar{u}\sin\bar{\theta}\sin\bar{\phi}\cos\bar{\psi} \\
&\quad -u\sin\theta\sin\phi\sin\psi\,\bar{u}\sin\bar{\theta}\sin\bar{\phi}\sin\bar{\psi} \\
&= u\bar{u}\cos\theta\cos\bar{\theta} - \sin\theta\sin\bar{\theta}(\cos\phi\cos\bar{\phi} \\
&\qquad + \sin\phi\cos\psi\sin\bar{\phi}\cos\bar{\psi} - \sin\phi\sin\psi\sin\bar{\phi}\sin\bar{\psi})
\end{aligned}$$

By the Law of Moduli, $\bar{\bar{u}} = u\bar{u}$ so that

$$\cos \bar{\bar{\theta}} = \cos \theta \cos \bar{\theta} - \sin \theta \sin \bar{\theta}(\cos \phi \cos \bar{\phi}$$
$$+ \sin \phi \cos \psi \sin \bar{\phi} \cos \bar{\psi} - \sin \phi \sin \psi \sin \bar{\phi} \sin \bar{\psi})$$
$$= \cos \theta \cos \bar{\theta} - \sin \theta \sin \bar{\theta} \cos \phi \cos \bar{\phi}$$
$$+ \sin \phi \sin \bar{\phi}(\cos \psi \cos \bar{\psi} - \sin \psi \sin \bar{\psi})$$
$$= \cos \theta \cos \bar{\theta} - \sin \theta \sin \bar{\theta} \cos \phi \cos \bar{\phi}$$
$$+ \sin \phi \sin \bar{\phi} \cos(\psi - \bar{\psi}) \ .$$

Consequently, by equation (12)

$$\cos \bar{\bar{\theta}} = \cos \theta \cos \bar{\theta} - \sin \theta \sin \bar{\theta} \cos \angle AOB.$$

To solve for $\cos \theta$ consider the following equation. We have

$$xp+yq+zr+ws=x(ax-by-cz-dw)+y(ay+bx+cw-dz)$$
$$+z(az-bw-cz+dy)+w(aw+bz-cy+dx)$$
$$=ax^2+ay^2+az^2+aw^2.$$

Hence,

$$xp+yq+zr+ws=a(x^2+y^2+z^2+w^2)$$
$$=u\bar{u}^2 \cos \theta$$

since $a = u \cos \theta$ and $\bar{u}^2 = x^2 + y^2 + z^2 + w^2$. Replacing the left member of the equation with spherical coordinate values yields

$$u\bar{u}^2 \cos \theta = \bar{u} \cos \bar{\bar{u}} \cos \bar{\bar{\theta}} + \bar{u} \sin \bar{\theta} \cos \bar{\phi}\, \bar{\bar{u}} \sin \bar{\bar{\theta}} \cos \bar{\bar{\theta}}$$
$$+ \bar{u} \sin \bar{\theta} \cos \bar{\psi}\, \bar{\bar{u}} \sin \bar{\bar{\theta}} \sin \bar{\bar{\phi}} \cos \bar{\bar{\psi}}$$
$$+ \bar{u} \sin \bar{\theta} \sin \bar{\phi} \sin \bar{\psi}\, \bar{\bar{u}} \sin \bar{\bar{\theta}} \sin \bar{\bar{\phi}} \sin \bar{\bar{\psi}}.$$
$$= u\bar{\bar{u}} \cos \bar{\theta} \cos \bar{\bar{\theta}} + \sin \bar{\theta} \sin \bar{\bar{\theta}}(\cos \bar{\phi} \cos \bar{\bar{\phi}}$$
$$+ \sin \bar{\phi} \cos \bar{\psi} \sin \bar{\bar{\phi}} \cos \bar{\bar{\psi}} + \sin \bar{\phi} \sin \bar{\psi} \sin \bar{\bar{\phi}} \sin \bar{\bar{\psi}})$$

and

$$\cos\theta = \cos\bar{\theta}\cos\bar{\bar{\theta}} + \sin\bar{\theta}\sin\bar{\bar{\theta}}\cos\bar{\phi}\cos\bar{\bar{\phi}}$$
$$+ \sin\bar{\phi}\sin\bar{\bar{\phi}}\cos(\bar{\psi} - \bar{\bar{\psi}}) \ .$$

Again, using equation (12), we have

$$\cos\theta = \cos\bar{\theta}\cos\bar{\bar{\theta}} + \sin\bar{\theta}\sin\bar{\bar{\theta}}\cos\angle BOC.$$

Similarly,

$$ap+bq+cr+ds=a(ax-by-cz-dw)+b(ay+bx+cw-dz)$$
$$+c(az-bw+cx+dy)+d(aw+bz-cy+dx)$$
$$=a^2x+b^2x+c^2x+d^2x$$

and $ap+bq+cr+ds=\bar{u}u^2 \cos\bar{\theta}$. Substituting the spherical coordinate values into the left side and simplifying yields

$$\cos\theta = \cos\theta\cos\bar{\bar{\theta}} + \sin\theta\sin\bar{\bar{\theta}}\cos\angle AOC.$$

Considering the spherical triangle ABC, the arcs AB, BC and AC are measured by the angles AOB, BOC, and AOC, respectively. Therefore, the equations

$$\cos\bar{\bar{\theta}} = \cos\theta\cos\bar{\theta} - \sin\theta\sin\bar{\theta}\cos\angle AOB$$
$$\cos\bar{\theta} = \cos\theta\cos\bar{\bar{\theta}} + \sin\theta\sin\bar{\bar{\theta}}\cos\angle AOC$$
$$\cos\theta = \cos\bar{\theta}\cos\bar{\bar{\theta}} + \sin\bar{\theta}\sin\bar{\bar{\theta}}\cos\angle BOC$$

can be expressed as

$$\cos\bar{\bar{\theta}} = \cos\theta\cos\bar{\theta} - \sin\theta\sin\bar{\theta}\cos\widehat{AB}$$
$$\cos\bar{\theta} = \cos\theta\cos\bar{\bar{\theta}} + \sin\theta\sin\bar{\bar{\theta}}\cos\widehat{AC}$$
$$\cos\theta = \cos\bar{\theta}\cos\bar{\bar{\theta}} + \sin\bar{\theta}\sin\bar{\bar{\theta}}\cos\widehat{BC}.$$

Letting $\bar{\bar{\theta}} = \pi - r$, $\bar{\theta} = \beta$ and $\theta = \alpha$ the above equations become

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\widehat{AB}$$

$$\cos\beta = -\cos\alpha\cos\gamma + \sin\alpha\sin\gamma\cos\widehat{AC}$$

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\widehat{BC}$$

which is just a restatement of the Law of Cosines for the angles of a spherical triangle (3:213). Consequently, we have that

$$\theta = \alpha,\ \overline{\theta} = \beta \text{ and } \overline{\overline{\theta}} = \pi - \gamma \tag{14}$$

where θ, $\overline{\theta}$ and $\overline{\overline{\theta}}$ are the amplitudes of (a,b,c,d), (x,y,z,w) and (p,q,r,s), respectively, and α, β, γ are the spherical angles determined by imag(a,b,c,d), imag(x,y,z,w) and imag(p,q,r,s), respectively. These relationships are illustrated by Figure 2.

The equations given by (14) may be summarized as follows. Given any two quaternions and their product a spherical triangle is determined on a unit sphere by the imaginary components of those quaternions such that

i) the amplitude of a factor of the product and the spherical angle determined by the imaginary component of that factor have the same measure, and

ii) the amplitude of the product and the spherical angle determined by the imaginary component of the product are supplementary.

Another interesting relationship can be found if we take the product (0,b,c,d)(0,y,z,w)=(p,q,r,s) and consider that $p=-by-cz-dw$ by equation (9) and $p=\bar{\bar{u}} \cos \bar{\bar{\theta}}$ by equation (11). Combining these equations we have

$$-by-cz-dw=\sqrt{p^2+q^2+r^2+s^2} \quad \cos \bar{\bar{\theta}}$$

and

$$\cos \bar{\bar{\theta}} = \frac{-by-cz-dw}{\sqrt{p^2+q^2+r^2+s^2}} .$$

By the Law of Moduli,

$$\sqrt{p^2+q^2+r^2+s^2} = \sqrt{b^2+c^2+d^2} \sqrt{y^2+z^2+w^2}.$$

Hence,

$$\cos \bar{\bar{\theta}} = \frac{-by-cz-dw}{\sqrt{b^2+c^2+d^2} \sqrt{y^2+z^2+w^2}} .$$

If we let α denote the angle between the vectors determined by $(0,b,c,d)$ and $(0,y,z,w)$, then applying equation (13) to the above equation we have

$$\cos \bar{\bar{\theta}} = -\cos \alpha .$$

Consequently,

$$\cos \bar{\bar{\theta}} = \cos(\pi - \alpha)$$

and we have

$$\bar{\bar{\theta}} = \pi - \alpha. \tag{15}$$

Equation (15) shows that the amplitude of the product is a function of the angle α between its factors. Notice that (14) and (15) together yield

$$\alpha = r.$$

Consider now the consistency of equation (15) with the fundamental identities

i) $i^2=j^2=k^2=-1$,

ii) $ij=k$, $jk=i$, $ki=j$.

If i) holds, then $\alpha = 0$ since the angle between any imaginary and itself is zero. According to (15), $\overline{\overline{\theta}} = \pi$, which is true since the product -1 is opposite the positive real unit vector. If ii) holds and the product is of the form $ij=k$, then $\alpha = \pi/2$ since i, j and k are perpendicular. By (15) $\overline{\overline{\theta}} = \pi/2$ which is true since all four axes of the quaternion coordinate system are perpendicular. Consequently, equations (14) and (15) along with Figure 2 help to describe in three space the four dimensional concept of a quaternion.

The algebraic product of two quaternions is given by equations (9). However, we know little about the geometrical interpretation of equations (9). If we consider quaternions of the form $(0,b,c,d)$, then the product of two such elements as given by (9) can be shown geometrically.

Given elements (a,b,c,d) and (t,p,q,r), let $\alpha = bi+cj+dk$ and $\beta = pi+qj+rk$. We will consider α and β as vectors since both are a linear combination of the unit vectors i,j,k. By equations (9) $\alpha\beta = -(bp+cq+dr)+i(cr-dg)+j(dp-br)+k(bq-cp)$ indicating that $\alpha\beta$ is the sum of a scalar and an imaginary. The imaginary member is a vector for the same reason that α and β are vectors. We will denote the scalar of $\alpha\beta$ by $S_{\alpha\beta}$ and the imaginary by $V_{\alpha\beta}$.

Since

$\beta\alpha = -(bp+cq+dr)+i(dq-cr)+j(br-dp)+k(cp-bq)$ we have that

$$S_{\alpha\beta} = S_{\beta\alpha} \tag{16}$$

and

$$V_{\alpha\beta} = -V_{\beta\alpha}. \tag{17}$$

From (16) and (17),

$$\alpha\beta + \beta\alpha = 2S_{\alpha\beta} \tag{18}$$

and

$$\alpha\beta - \beta\alpha = 2V_{\alpha\beta}. \tag{19}$$

Equations (16) and (17) can be verified geometrically with the help of Figures 3 and 4, pages 33 and 36.

When considering the product $\alpha\beta$, from a geometrical point of view, one must first consider the angle of inclination between α and β. Denoting this angle by ϕ, it follows that $0 \leq \phi \leq \pi$ and one of the following is true:

i) $\phi < \pi/2$,

ii) $\phi = \pi/2$, or

iii) $\phi > \pi/2$.

Figures 3 and 4, pages 33 and 36, correspond to cases i) and iii), respectively.

Refering to Figure 3, if $\phi < \pi/2$, then the product $\alpha\beta$ can be expressed as $\alpha(\beta_1+\beta_2)$ since β is the vector sum of β_1 and β_2.

Hence, $\alpha(\beta_1+\beta_2) = \alpha\beta_1 + \alpha\beta_2$.

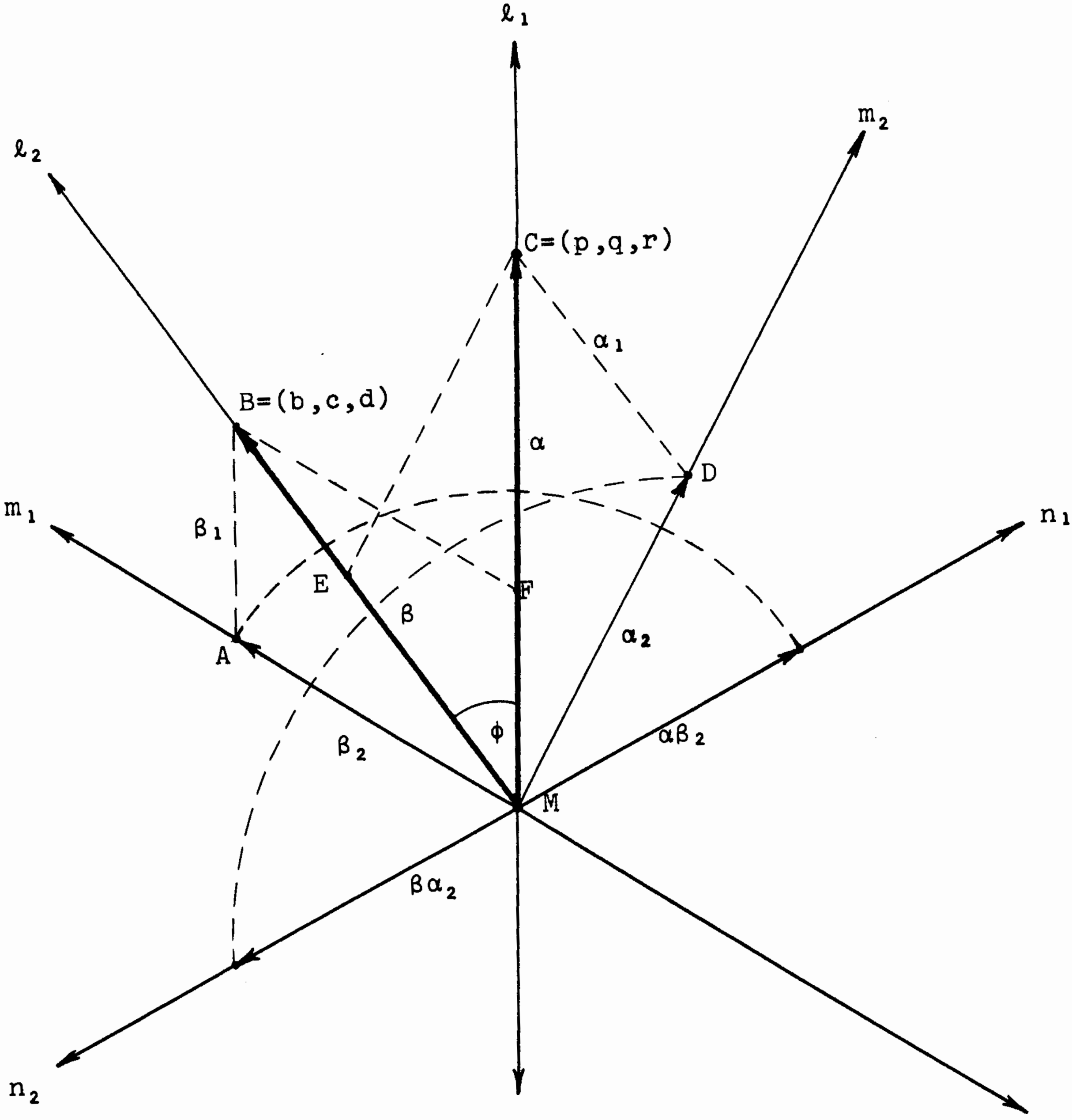

FIGURE 3

THE GEOMETRICAL REPRESENTATION FOR THE REAL AND IMAGINARY COMPONENTS OF A PRODUCT WHEN THE ANGLE BETWEEN ITS FACTORS IS ACUTE

From Figure 3, page 33, we see that β_1 and β_2 are positioned so that β_2 is perpendicular to the ℓ_1-axis with β_1 parallel to the ℓ_1-axis. Similarly, α_2 is perpendicular to ℓ_2 with α_1 parallel to ℓ_2. Note that ℓ_1, ℓ_2, m_1, and m_2 are coplanar. Letting $|\alpha|$ represent the magnitude of vector α and taking i to be the unit vector of α we have that $\alpha = |\alpha|i$. Since α and β_1 are parallel, $\beta_1 = |\beta_1|i$. Consequently,

$$\alpha\beta_1 = -|\alpha||\beta_1|. \tag{20}$$

Similarly, if j is the unit vector of β, then $\beta = |\beta|j$ and $\alpha_1 = |\alpha_1|j$ since β and α_1 are parallel. Thus we have

$$\beta\alpha_1 = -|\beta||\alpha_1|. \tag{21}$$

Now consider similar triangles MBF and MCE in Figure 3, page 33. Taking the ratios of corresponding sides yields

$$\frac{|\beta_1|}{|\beta|} = \frac{|\alpha_1|}{|\alpha|} \quad \text{or} \quad |\alpha||\beta_1| = |\beta||\alpha_1|.$$

Combining these results with equations (20) and (21), we have $\alpha\beta_1 = \beta\alpha_1$ since $\alpha\beta_1$ and $\beta\alpha_1$ have the same magnitude and direction. Since $\alpha\beta_1$ and $\beta\alpha_1$ are scalar or real quantities we let

$$S_{\alpha\beta} = \alpha\beta_1 \text{ and}$$

$$S_{\beta\alpha} = \beta\alpha_1.$$

Therefore, $S_{\alpha\beta} = S_{\beta\alpha}$ for $0 < \phi < \pi/2$. When $\phi = 0$, $\alpha_1 = \beta_1 = \alpha_1 = \beta$ so that

$$\alpha\beta_1 = \beta\alpha_1 = -|\alpha|^2$$

and

$$S_{\alpha\beta} = S_{\beta\alpha} \text{ for } \phi < \pi/2.$$

If $\phi = \pi/2$, then $\beta_1 = \alpha_1 = 0$ so that $\alpha\beta_1 = \beta\alpha_1 = 0$ and

$$S_{\alpha\beta} = S_{\beta\alpha} \text{ for } \phi = \pi/2.$$

If $\pi/2 < \phi < \pi$, then Figure 4, page 36, shows that ℓ_1, ℓ_2, m_1 and m_2 are coplanar. Further, ℓ_1 is perpendicular to m_1 and ℓ_2 is perpendicular to m_2. Consequently, ϕ_1 and ϕ_2 are complements of AMD and $\phi_1 = \phi_2$.

Let i be the unit vector of α. Then, the unit vector for β_1 is $-i$ since α and β_1 are opposite. Thus

$$\alpha = |\alpha| i \text{ and } \beta_1 = -|\beta_1| i$$

so that

$$\alpha\beta_1 = |\alpha||\beta_1|. \tag{22}$$

Similarly, let j be the unit vector of β. Then $\beta = |\beta| j$ and $\alpha_1 = -|\alpha_1| j$ yielding

$$\beta\alpha_1 = |\beta||\alpha_1|. \tag{23}$$

Since $\phi_1 = \phi_2$, triangles ABM and CDM are similar and

$$\frac{|\alpha_1|}{|\alpha|} = \frac{|\beta_1|}{|\beta|} \text{ or } |\beta||\alpha_1| = |\alpha||\beta_1|.$$

Combining these results with equations (22) and (23) we have

$$S_{\alpha\beta} = S_{\beta\alpha} \text{ for } \pi/2 < \phi < \pi$$

If $\phi = \pi$, then $\alpha = \alpha_1$ and $\beta = \beta_1$ so that $|\alpha||\beta| = |\beta||\alpha| = |\alpha||\beta_1| = |\beta||\alpha_1|$. Let i denote the unit vector of α. Then $-i$ denotes the unit vector of β since α and β are opposite. We have

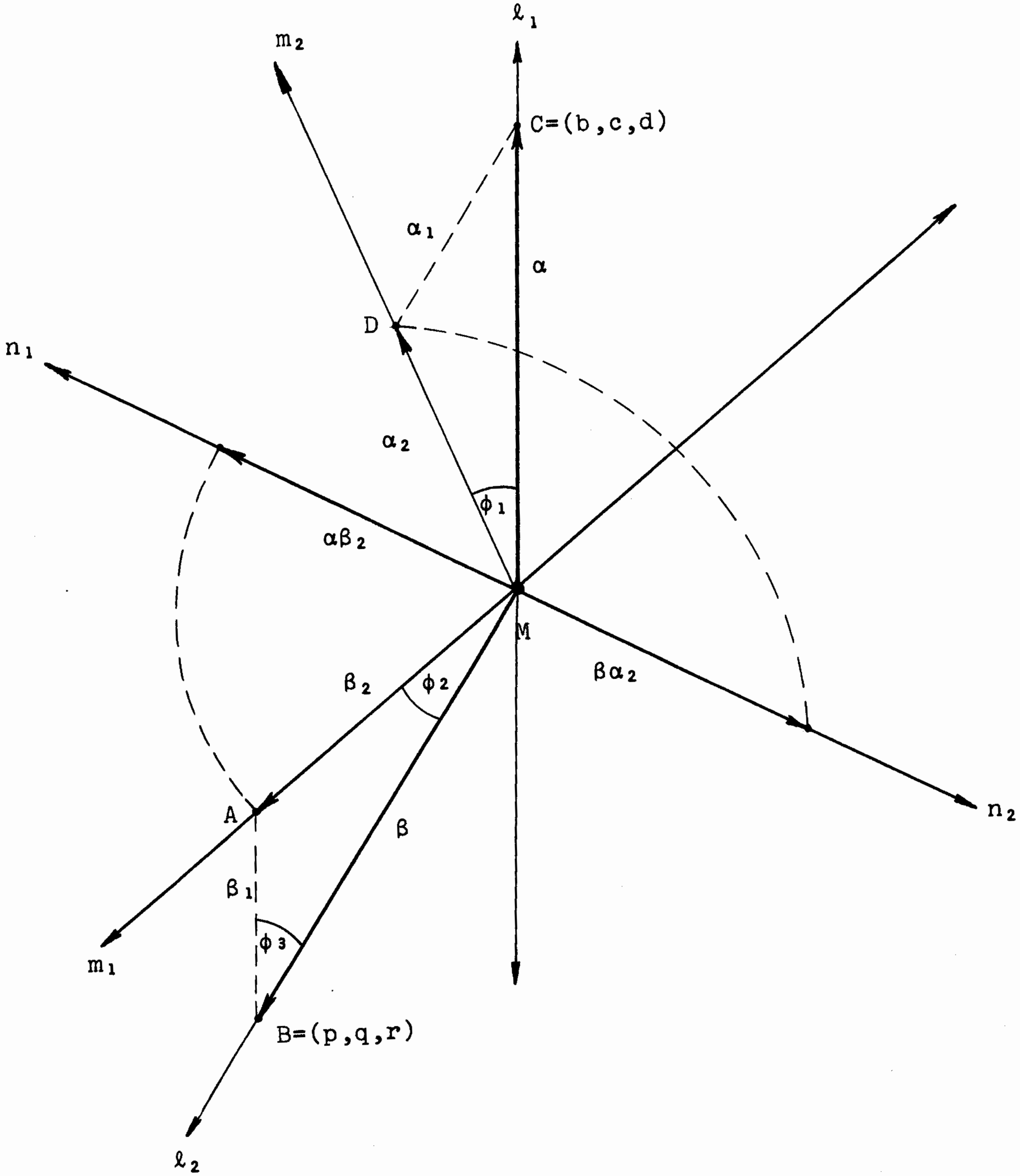

FIGURE 4

THE GEOMETRICAL REPRESENTATION FOR THE REAL AND IMAGINARY COMPONENTS OF A PRODUCT WHEN THE ANGLE BETWEEN ITS FACTORS IS OBTUSE

$$\alpha\beta_1 = -|\alpha||\beta_1|i^2 = |\alpha||\beta_1|$$

and

$$\beta\alpha_1 = -|\beta||\alpha_1|i^2 = |\beta||\alpha_1|$$

yielding

$$S_{\alpha\beta} = S_{\beta\alpha} \text{ for } \phi = \pi.$$

Therefore,

$$S_{\alpha\beta} = S_{\beta\alpha} \text{ for all } \phi > \pi/2.$$

In general, for any two quaternions, α, β of the form $\alpha=(0,b,c,d)$, $\beta=(0,p,q,r)$, it is always the case that $S_{\alpha\beta} = S_{\beta\alpha}$. Notice that by using equations (9) we can show $S_{\alpha\beta} = S_{\beta\alpha}$ for any two quaternions. However, the geometrical interpretation in four space is difficult to illustrate with a figure.

Consider now the relationship between $\alpha\beta_2$ and $\beta\alpha_2$. Again let ϕ represent the angle between α and β. Then, either

i) $\phi < \pi/2$,

ii) $\phi = \pi/2$, or

iii) $\phi > \pi/2$.

If $0 < \phi < \pi/2$, then Figure 3, page 33, indicates that α and β_2 are perpendicular as are β and α_2. Let i and j denote the unit vectors of α and β_2, respectively. Then, $\alpha = |\alpha|i$ and $\beta_2 = |\beta_2|j$ so that

$$\alpha\beta_2 = |\alpha||\beta_2|k \tag{24}$$

where k is the unit vector of $\alpha\beta$ determined by the Rule of Rotation. Let i' and j' be the unit vectors of β and α, respectively. Then

$$\beta = |\beta| i' \text{ and } \alpha_2 = |\alpha_2| j'$$

so that

$$\beta\alpha_2 = |\beta||\alpha_2| k' \tag{25}$$

where k' is the unit vector of $\beta\alpha_2$ determined by the Rule of Rotation. Figure 3, page 33, along with the Rule of Rotation, shows that $k=-k'$. Using this fact with equations (24) and (25) yields

$$\alpha\beta_2 = |\alpha||\beta_2| k$$

$$\beta\alpha_2 = -|\beta||\alpha_2| k$$

indicating $\alpha\beta_2$ and $\beta\alpha_2$ have opposite directions. From Figure 3, page 33, triangles ABM and CDM are similar which implies

$$\frac{|\beta_2|}{|\beta|} = \frac{|\alpha_2|}{|\alpha|} \text{ or } |\alpha||\beta_2| = |\beta||\alpha_2|.$$

Thus, $\alpha\beta_2 = -\beta\alpha_2$ for $0 < \phi < \pi/2$. When $\phi = 0$, $\beta_2 = \alpha_2 = 0$ so that $\alpha\beta_2 = \beta\alpha_2 = 0$ and we have that $\alpha\beta_2 = -\beta\alpha_2$ for all $\phi < \pi/2$.

If $\phi = \pi/2$, then $\beta_2 = \beta$ and $\alpha_2 = \alpha$ yielding $|\alpha||\beta| = |\beta||\alpha| = |\alpha||\beta_2| = |\beta||\alpha_2|$. Equations (8) indicate that $\alpha\beta_2$ and $\beta\alpha_2$ have opposite directions. Therefore, $\alpha\beta_2 = -\beta\alpha_2$ for $\phi = \pi/2$.

If $\pi/2 < \phi < \pi$, then let i, j, k, i', j', k' denote the same unit vectors as in the case involving $0 < \phi < \pi/2$. Using Figure 4, page 36, and these conventions we have $\alpha = |\alpha|i$, $\beta_2 = |\beta_2|j$ so that $\alpha\beta_2 = |\alpha||\beta_2|k$. Similarly, $\beta = |\beta|i'$, $\alpha_2 = |\alpha_2|j'$ and $\beta\alpha_2 = |\beta||\alpha_2|k'$. From Figure 4, page 36, $\phi_1 = \phi_2$ since both are complements of the same angle. This indicates triangles ABM and CDM are similar where

$$\frac{|\beta_2|}{|\beta|} = \frac{|\alpha_2|}{|\alpha|} \quad \text{or} \quad |\alpha||\beta_2| = |\beta||\alpha_2|.$$

Combining this with $k=-k'$ we obtain $\alpha\beta_2 = -\beta\alpha_2$ for $\pi/2<\phi<\pi$.

Finally, if $\phi = \pi$, then $\alpha_2 = \beta_2 = 0$ and $\alpha\beta_2 = \beta\alpha_2 = 0$. Consequently, $\alpha\beta_2 = -\beta\alpha_2$ for all $\phi > \pi/2$ and we have that in general, $\alpha\beta_2 = -\beta\alpha_2$ for all α, β of the form $\alpha = (0,b,c,d)$, $\beta = (0,p,q,r)$. Letting $V_{\alpha\beta} = \alpha\beta_2$ and $V_{\beta\alpha} = \beta\alpha_2$, substitution yields

$$V_{\alpha\beta} = -V_{\beta\alpha}.$$

Equations (9) can be used to show that $V_{\alpha\beta} = -V_{\beta\alpha}$ for any two quaternions although it is difficult to illustrate this with a figure.

As has been suggested, the quantities $S_{\alpha\beta}$ and $V_{\alpha\beta}$ are a function of the angle between α and β. We will now derive an expression for $S_{\alpha\beta}$ and $V_{\alpha\beta}$ in terms of this angle.

Let ϕ represent the angle between two vectors α and β determined by the points (0,b,c,d) and (0,p,q,r), respectively. Figure 3, page 33, shows that $\cos \phi = \dfrac{|\beta_1|}{|\beta|}$ when $0 < \phi < \pi/2$. Thus, $|\beta_1| = |\beta| \cos \phi$.

When $0 < \phi < \pi/2$, we also have $S_{\alpha\beta} = -|\alpha||\beta_1|$ so that $S_{\alpha\beta} = -|\alpha||\beta| \cos \phi$ for $0 < \phi < \pi/2$.

If $\pi/2 < \phi < \pi$, then $\cos \phi = -\cos(\pi - \phi)$. From Figure 4, page 36, $\phi_3 = \pi - \phi$, since β_1 and ℓ_1 are parallel. Note that $\cos \phi_3 = \dfrac{|\beta_1|}{|\beta|}$. Combining these results we obtain $\cos \phi_3 = \cos(\pi - \phi) = -\cos \phi = \dfrac{|\beta_1|}{|\beta|}$ implying $|\beta_1| = -|\beta| \cos \phi$. When $\pi/2 < 0 < \pi$, we have previously calculated that $S_{\alpha\beta} = |\alpha||\beta_1|$. Using this result we obtain $S_{\alpha\beta} = -|\alpha||\beta| \cos \phi$ for $\pi/2 < \phi < \pi$.

For $\phi = 0$, $\phi = \pi$ and $\phi = \pi/2$, $S_{\alpha\beta} = -|\alpha||\beta| \cos \phi$ is consistent with equations (8). Hence, for all α, β as defined

$$S_{\alpha\beta} = -|\alpha||\beta| \cos \phi. \tag{26}$$

Consider now the relationship between $V_{\alpha\beta}$ and angle ϕ. As already discussed, $V_{\alpha\beta}$ is perpendicular to the plane containing α and β. Let r denote the unit vector of $V_{\alpha\beta}$ so that $V_{\alpha\beta} = |V_{\alpha\beta}|r$.

If $0 < \phi < \pi/2$, then Figure 3, page 33, indicates $\sin \phi = \dfrac{|\beta_2|}{|\beta|}$ which gives $|\beta_2| = |\beta| \sin \phi$. Since

$V_{\alpha\beta} = \alpha\beta_2$ we have $|V_{\alpha\beta}| = |\alpha\beta_2| = |\alpha||\beta_2|$ by the Law of Moduli. Therefore, $|V_{\alpha\beta}| = |\alpha||\beta| \sin\phi$. Since $V_{\alpha\beta} = |V_{\alpha\beta}|r$ we obtain

$V_{\alpha\beta} = |\alpha||\beta|r \sin\phi$ for $0 < \phi < \pi/2$.

If $\pi/2 < \phi < \pi$, then Figure 4, page 36, shows that $\phi_3 = \pi - \phi$ and $\sin\phi_3 = \dfrac{|\beta_2|}{|\beta|} = \sin(\pi - \phi) = \sin\phi$.

Substitution yields $|V_{\alpha\beta}| = |\alpha||\beta| \sin\phi$ so that $V_{\alpha\beta} = |\alpha||\beta|r \sin\phi$ for $\pi/2 < \phi < \pi$. When $\phi = 0$, $\phi = \pi/2$ or $\phi = \pi$ the results of the above equation are consistent with equations (8). Hence,

$$V_{\alpha\beta} = |\alpha||\beta|r \sin\phi \tag{27}$$

for all α, β as previously defined. With the use of equations (20)-(27) it can be shown that

$$\alpha\beta + \beta\alpha = -2|\alpha||\beta| \cos\phi \tag{28}$$

and

$$\alpha\beta - \beta\alpha = 2|\alpha||\beta|r \sin\phi. \tag{29}$$

If $\alpha = \beta$, then equation (28) yields $|\alpha| = \sqrt{-\alpha^2}$ which is consistent with equations (8).

If $\alpha\beta + \beta\alpha = 0$, then equation (28) implies α and β are perpendicular.

If $\alpha\beta - \beta\alpha = 0$, then equation (29) implies α and β are either opposite or have the same direction.

As a result of these rules, the magnitude of the resultant of n vectors may be calculated as described below.

Let α_1, α_2, α_3, ..., α_n denote vectors with magnitudes, A_1, A_2, A_3, ..., A_n, respectively. If β denote the resultant of vectors α_1, α_2, α_3, ..., α_n then

$$\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots + \alpha_n$$

and

$$\beta^2=\alpha_1^2+\alpha_2^2+\ldots\alpha_n^2+\alpha_1\alpha_2+\alpha_2\alpha_1+\ldots\alpha_1\alpha_n+\alpha_n\alpha_1$$
$$+\alpha_2\alpha_3+\alpha_3\alpha_2+\ldots+\alpha_2\alpha_n+\alpha_n\alpha_2+\alpha_3\alpha_4+\alpha_4\alpha_3+\ldots+\alpha_3\alpha_n+\alpha_n\alpha_3$$
$$+\ldots+\alpha_{n-1}\alpha_n+\alpha_n\alpha_{n-1}.$$

Since $|\beta| = \sqrt{-\beta^2}$ and $\alpha\beta + \beta\alpha = -2|\alpha||\beta| \cos \phi$ it follows that

$$-|\beta|^2=-A_1^2-A_2^2-\ldots-A_n^2-2A_1A_2\cos(A_1,A_2)-2A_1A_3\cos(A_1,A_3)-\ldots$$
$$-2A_1A_n\cos(A_1,A_n)-\ldots-2A_{n-1}A_n\cos(A_{n-1},A_n)$$

where (A_1,A_2) represents the angle between vectors α_1 and α_2.

Simplification yields

$$|\beta|= \sqrt{A_1^2+A_2^2+\ldots+A_n^2+2\ A_1A_2\cos(A_1,A_2)+A_1A_3\cos(A_1,A_3)+\ldots+A_{n-1}A_n\cos(A_{n-1},A_n)}\ .$$

This expression is equivalent to $|\beta| = \sqrt{(\Sigma x)^2+(\Sigma y)^2+(\Sigma z)^2}$ where x,y,z represent the vector components associated with the X, Y and Z axes in three space.

As another application of expressions (20)-(29), let α and β be three dimensional vectors determined by the points (0,a,b,c) and (0,x,y,z), respectively. Given two such vectors, one could rotate β around α to a new position β'. Letting m denote the magnitude of the rotation, it is possible to express β' in terms of m, α and β.

Using Figure 5, page 45, let α denote the axis about which β will rotate m degrees to a new position β'. Vector β has components β_1 and β_2 such that $\beta = \beta_1 + \beta_2$. We have that β_1 is coincidental with α and β_2 is perpendicular to α. Since $\alpha\beta + \beta\alpha = 2S_{\alpha\beta} = 2\alpha\beta_1$, it follows that $-\frac{1}{2}\alpha(\alpha\beta + \beta\alpha) = -\frac{1}{2}\alpha(2\alpha\beta_1)$ and

$$\beta_1 = \frac{1}{2}(\beta - \alpha\beta\alpha). \tag{30}$$

Similarly,

$$\beta_2 = \frac{1}{2}(\beta + \alpha\beta\alpha). \tag{31}$$

Since β_1 is unchanged by the rotation,

$$\beta_1 = \beta_1' = \frac{1}{2}(\beta - \alpha\beta\alpha). \tag{32}$$

Let c_1 and c_2 denote the components of β_2' such that $c_1 + c_2 = \beta_2'$. By the nature of the rotation, β_2 and β_2' have the same magnitude. That is, $|\beta_2| = |\beta_2'|$. Figure 5, page 45, shows that

$$\cos m = \frac{|c_2|}{|\beta_2'|} \text{ or } |c_2| = |\beta_2'| \cos m.$$

Therefore, $|c_2| = |\beta_2| \cos m$. Using equation (31) $|c_2| = |\frac{1}{2}(\beta + \alpha\beta\alpha)| \cos m$. Since β_2 and c_2 are coincidental, we have that

$$c_2 = \frac{1}{2}(\beta + \alpha\beta\alpha) \cos m. \tag{33}$$

Similarly, $\sin m = \frac{|c_1|}{|\beta_2'|}$ or $|c_1| = |\beta_2'| \sin m$. Since $\beta_2 = \frac{1}{2}(\beta + \alpha\beta\alpha)$, substitution yields $\alpha\beta_2 = \alpha\ \frac{1}{2}(\beta + \alpha\beta\alpha) = \frac{1}{2}(\alpha\beta - \beta\alpha)$. Using the Rule of Rotation and Figure 5, page 45, we see that $\alpha\beta_2 = c_1$. Therefore,

$$c_1 = \frac{1}{2}(\alpha\beta - \beta\alpha). \tag{34}$$

Noting that $\beta' = \beta_1' + \beta_2'$, $\beta_2' = c_1 + c_2$ and $\beta_1 = \beta_1'$, substitution yields $\beta' = \beta_1 + c_1 + c_2$. Into this equation we substitute equations (32)-(34) and obtain $\beta' = \frac{1}{2}(\beta-\alpha\beta\alpha)+\frac{1}{2}(\alpha\beta-\beta\alpha)\sin m + \frac{1}{2}(\beta+\alpha\beta\alpha)\cos m$. Using the identities $\cos m = 2\cos^2\frac{m}{2} - 1 = 1 - 2\sin^2\frac{m}{2}$ and $\sin m = 2\sin\frac{m}{2}\cos\frac{m}{2}$ we obtain

$$\begin{aligned}
\beta' &= \tfrac{1}{2}(\beta-\alpha\beta\alpha) + \tfrac{1}{2}(2\sin\tfrac{m}{2}\cos\tfrac{m}{2})(\alpha\beta-\beta\alpha) \\
&\quad + \tfrac{1}{2}\beta\cos m + \tfrac{1}{2}\alpha\beta\alpha\cos m \\
&= \tfrac{1}{2}(\beta-\alpha\beta\alpha) + \sin\tfrac{m}{2}\cos\tfrac{m}{2}(\alpha\beta-\beta\alpha) \\
&\quad + \tfrac{1}{2}\beta(2\cos^2\tfrac{m}{2} - 1) + \tfrac{1}{2}\alpha\beta\alpha(1 - 2\sin^2\tfrac{m}{2}) \\
&= \tfrac{1}{2}\beta - \tfrac{1}{2}\alpha\beta\alpha + \sin\tfrac{m}{2}\cos\tfrac{m}{2}(\alpha\beta-\beta\alpha) \\
&\quad + \beta\cos^2\tfrac{m}{2} - \tfrac{1}{2}\beta + \tfrac{1}{2}\alpha\beta\alpha - \alpha\beta\alpha\sin^2\tfrac{m}{2} \\
&= \beta\cos^2\tfrac{m}{2} - \alpha\beta\alpha\sin^2\tfrac{m}{2} + \sin\tfrac{m}{2}\cos\tfrac{m}{2}(\alpha\beta-\beta\alpha) \\
&= \beta\cos^2\tfrac{m}{2} - \alpha\beta\alpha\sin^2\tfrac{m}{2} + \alpha\beta\sin\tfrac{m}{2}\cos\tfrac{m}{2} \\
&\quad - \beta\alpha\sin\tfrac{m}{2}\cos\tfrac{m}{2}
\end{aligned}$$

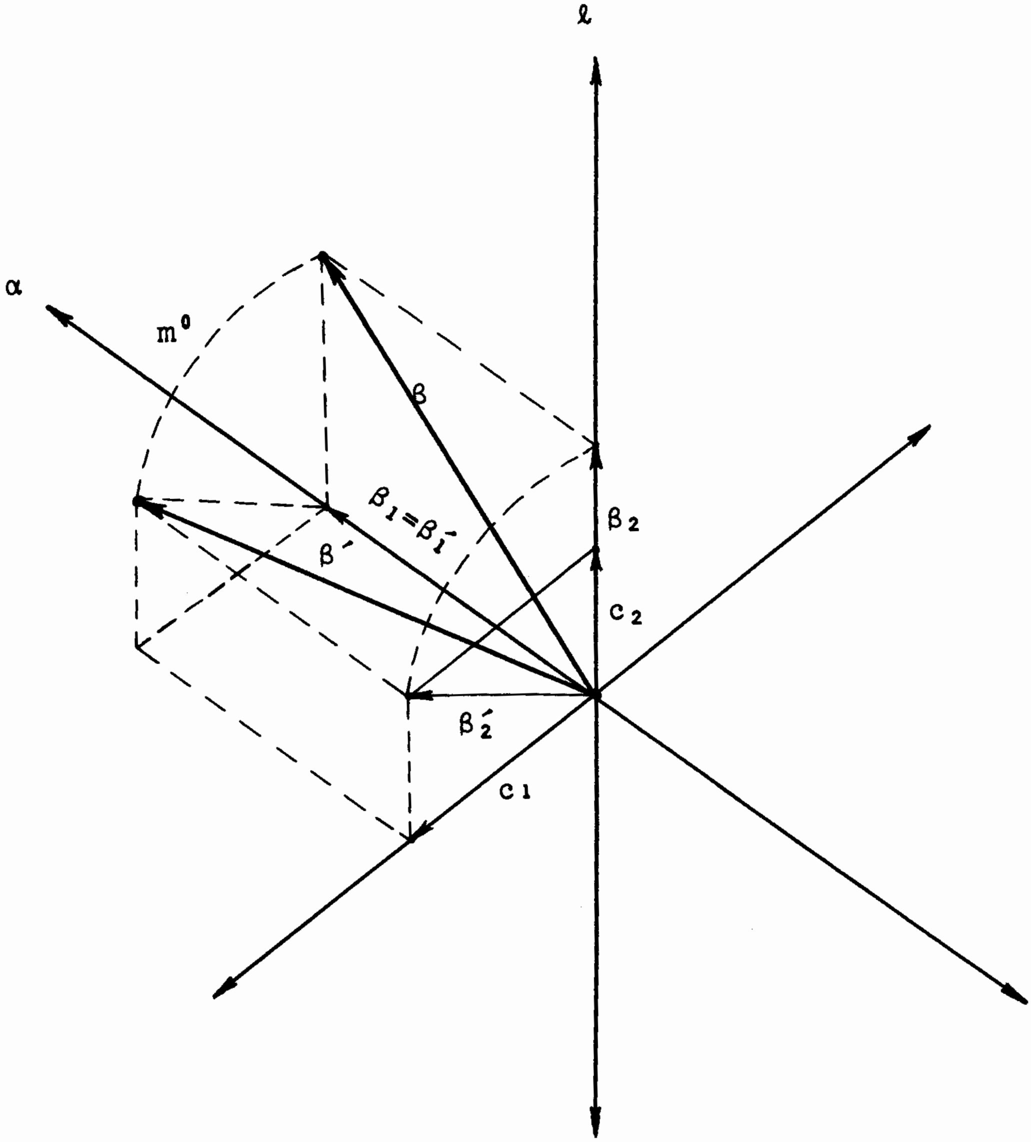

FIGURE 5

THE ROTATION OF VECTOR β ABOUT
VECTOR α THROUGH AN ANGLE
OF m DEGREES

$$= \beta\cos^2\frac{m}{2} + \alpha\beta\sin\frac{m}{2}\cos\frac{m}{2} - \alpha\beta\alpha\sin^2\frac{m}{2}$$

$$- \beta\alpha\sin\frac{m}{2}\cos\frac{m}{2}$$

$$= (\cos^2\frac{m}{2} + \alpha\sin\frac{m}{2}\cos\frac{m}{2})\beta-(\alpha\sin^2\frac{m}{2} + \sin\frac{m}{2}\cos\frac{m}{2})\beta\alpha$$

$$= \cos\frac{m}{2}(\cos\frac{m}{2} + \alpha\sin\frac{m}{2})\beta-\sin\frac{m}{2}(\alpha\sin\frac{m}{2} + \cos\frac{m}{2})\beta\alpha.$$

Therefore, $\beta' = (\cos\frac{m}{2} + \alpha\sin\frac{m}{2})(\beta\cos\frac{m}{2} - \beta\alpha\sin\frac{m}{2})$

which can be expressed as

$$\beta' = (\cos\frac{m}{2} + \alpha\sin\frac{m}{2})\beta(\cos\frac{m}{2} - \alpha\sin\frac{m}{2}). \quad (35)$$

Notice that whenever $m = 0$, Figure 5, page 45, indicates $\beta' = \beta$ which is consistent with equation (35).

CHAPTER V

QUATERNIONS AS AN EXTENSION OF THE COMPLEX NUMBERS

In order to investigate the connection between the complex numbers and quaternions, we use the power series expansion for e^x, cos x and sin x.

For any complex number of the form (0,x), one can easily show that

$$e^{(0,x)} = \cos x + i \sin x \tag{36}$$

or equivalently

$$e^{ix} = \cos x + i \sin x.$$

Since (0,x,y,z) can be thought of as a more general form of (0,x), we would hope to find a more general form for $e^{(0,x)}$ by considering $e^{(0,x,y,z)}$. We have

$$e^{(0,x)} = e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots.$$

Therefore, we let

$$e^{(0,x,y,z)} = e^{ix+jy+kz}$$
$$=1+(ix+jy+kz) + \frac{(ix+jy+kz)^2}{2!} + \frac{(ix+jy+kz)^3}{3!}$$
$$+\frac{(ix+jy+kz)^4}{4!} + \cdots.$$

Application of definition (9) yields

$$(xi+yj+zk)^2 = -(x^2+y^2+z^2)$$

$$(xi+yj+zk)^3 = -(x^2+y^2+z^2)(xi+yj+zk)$$

$$(xi+yj+zk)^4 = (x^2+y^2+z^2)^2$$

$$(xi+yj+zk)^5 = (x^2+y^2+z^2)^2(xi+yj+zk)$$

Using these equations we substitute into the equation for $e^{(xi+yj+zk)}$ and obtain

$$e^{(xi+yj+zk)} = 1+(xi+yj+zk)-\frac{(x^2+y^2+z^2)}{2!}-\frac{(x^2+y^2+z^2)(xi+yj+zk)}{3!}$$

$$+\frac{(x^2+y^2+z^2)^2}{4!}+\frac{(x^2+y^2+z^2)(xi+yj+zk)}{5!} - \cdots$$

$$= 1-\frac{(x^2+y^2+z^2)}{2!}+\frac{(x^2+y^2+z^2)^2}{4!}-\frac{(x^2+y^2+z^2)^3}{6!} - \cdots$$

$$+ (xi+yj+zk)-\frac{(x^2+y^2+z^2)(xi+yj+zk)}{3!}$$

$$+\frac{(x^2+y^2+z^2)^2(xi+yj+zk)}{5!} - \cdots$$

$$= 1-\frac{\sqrt{x^2+y^2+z^2}^{\,2}}{2!}+\frac{\sqrt{x^2+y^2+z^2}^{\,4}}{4!}+\frac{\sqrt{x^2+y^2+z^2}^{\,6}}{6!} + \cdots$$

$$+\frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}\left(\sqrt{x^2+y^2+z^2}-\frac{\sqrt{x^2+y^2+z^2}^{\,3}}{3!}+\frac{\sqrt{x^2+y^2+z^2}^{\,5}}{5!} - \cdots\right).$$

Since $\cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots$ and

$\sin a = a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \cdots$, we let

$a = \sqrt{x^2+y^2+z^2}$ and obtain

$$e^{(0,x,y,z)}=\cos\sqrt{x^2+y^2+z^2}+\frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}\sin\sqrt{x^2+y^2+z^2}. \qquad (37)$$

Notice that equation (37) reduces to equation (36) when $y = z = 0$, and that both $e^{(0,x)}$ and $e^{(0,x,y,z)}$ have a modulus of one. This is true since $\cos^2 x + \sin^2 x = 1$ and $\cos^2\sqrt{x^2+y^2+z^2} + \sin^2\sqrt{x^2+y^2+z^2} = 1$. Comparing equations (36) and (37) we see that i in equation (36) denotes the principal second root of negative one, thus, we are prompted to consider if the same is true of $\frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$ in equation (38). Notice that

$$\left(\frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}\right)^2 = \frac{-(x^2+y^2+z^2)}{x^2+y^2+z^2} = -1.$$

Therefore, $\frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$ represents a more general form for the principal second root of negative one provided $x^2+y^2+z^2 \neq 0$.

From equations (11), we let

$$u = \sqrt{x^2+y^2+z^2}$$

$$x = u \cos \phi,$$

$$y = u \sin \phi \cos \psi,$$

$$z = u \sin \phi \sin \psi,$$

represent the spherical coordinates of $(0,x,y,z)$. Substituting these quantities into equation (37), we obtain

$$e(0,x,y,z) = e^{iu \cos\phi+ju \sin\phi\cos\psi+ku \sin\phi\sin\psi}$$

$$= \cos\sqrt{u^2\cos^2\phi+u^2\sin^2\phi\cos^2\psi+u^2\sin^2\phi\sin^2\psi} + \frac{u(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)}{\sqrt{u^2\cos^2\phi+u^2\sin^2\phi\cos^2\psi+u^2\sin^2\phi\sin^2\psi}}$$

$$\sin\sqrt{u^2\cos^2\phi+u^2\sin^2\phi\cos^2\psi+u^2\sin^2\phi\sin^2\psi}$$

$$= \cos\sqrt{u^2\cos^2\phi+u^2\sin^2\phi(\cos^2\psi+\sin^2\psi)} + \frac{u(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)}{\sqrt{u^2\cos^2\phi+u^2\sin^2\phi(\cos^2\psi+\sin^2\psi)}}$$

$$\sin\sqrt{u^2\cos^2\phi+u^2\sin^2\phi(\cos^2\psi+\sin^2\psi)}.$$

Letting $r = i \cos \phi + j \sin \phi \cos \psi + k \sin \phi \sin \psi$, the above equation reduces to

$$e^{ur} = \cos u + r \sin u. \qquad (39)$$

Since $x = u \cos \phi$,

$y = u \sin \phi \cos \psi$,

$z = u \sin \phi \sin \psi$,

setting $\phi = 0$ would reduce $(0,x,y,z)$ to $(0,x,0,0)$ which is a more general form for $(0,x)$. Therefore, setting $\phi = 0$ should reduce equation (39) to complex form if it is a more general form for equation (36). Notice that when $\phi = 0$ we have $r = i$ and $u = x$ so that equation (39) becomes

$$e^{ix} = \cos x + i \sin x.$$

Further,

$$\begin{aligned} r^2 &= (i \cos \phi + j \sin \phi \cos \psi + k \sin \phi \sin \psi)^2 \\ &= i^2\cos^2\phi + ij\cos\phi\sin\phi\cos\psi + ik\cos\phi\sin\phi\sin\psi \\ &\quad + j^2\sin^2\phi\cos^2\psi + ji\cos\phi\sin\phi\cos\psi + jk\sin^2\phi\cos\psi\sin\psi \\ &\quad + ki\cos\phi\sin\phi\sin\psi + kj\sin^2\phi\cos\psi\sin\psi + k^2\sin^2\phi\sin^2\psi. \end{aligned}$$

Using equations (8), we obtain

$$\begin{aligned} r^2 &= i^2\cos^2\phi + k\cos\phi\sin\phi\cos\psi - j\cos\phi\sin\phi\sin\psi \\ &\quad + j^2\sin^2\phi\cos^2\psi - k\cos\phi\sin\phi\cos\psi + i\sin\phi\cos\psi\sin\psi \\ &\quad + j\cos\phi\sin\phi\sin\psi - i\sin^2\phi\cos\psi\sin\psi + k^2\sin^2\phi\sin^2\psi. \\ &= i^2\cos^2\phi + j^2\sin^2\phi\cos^2\phi + k^2\sin^2\phi\sin^2\psi \\ &= -\cos^2\phi + \sin^2\phi(\cos^2\psi + \sin^2\psi) \\ &= -1. \end{aligned}$$

Therefore, just as $i^2 = -1$ in the complex plane we have

$$(i \cos \phi + j \sin \phi \cos \psi + k \sin \phi \sin \psi)^2 = -1 \qquad (40)$$

in the system of quaternions. Notice that (40) holds true for all values of ϕ and ψ.

Considering quaternions of the form $(0,x,y,z)$, we have $(0,x,y,z)^2 = -(x^2+y^2+z^2)$. Thus, $(0,x,y,z)$ is a second root of negative one whenever $x^2+y^2+z^2=1$. Geometrically speaking, the points $(0,x,y,z)$ which represent the second roots of negative one are those points on a unit sphere whose origin is $(0,0,0)$ in the i,j,k coordinate system. In particular, the intersections of the i-axis with the sphere represent the second roots of negative one associated with the complex number system.

Any point (a,b) can be equivalently written in the form $m(\cos x + i \sin x)$. Therefore, we would hope to find a similar expression to represent the quaternion (a,b,c,d).

We have $(a,b,c,d)=a+bi+cj+dk$ and by equation (11)

$$a = u \cos \theta, \qquad c = u \sin \theta \sin \phi \cos \psi,$$

$$b = u \sin \theta \cos \phi, \qquad d = u \sin \theta \sin \phi \sin \psi.$$

Combining these five equations we obtain

$$\begin{aligned}(a,b,c,d) &= u \cos \theta + iu \sin \theta \cos \phi + ju \sin \theta \sin \phi \cos \psi \\ &\quad + ku \sin \theta \sin \phi \sin \psi \\ &= u \cos \theta + (i \cos \phi + j \sin \phi \cos \psi + k \sin \phi \sin \psi) \sin \theta\end{aligned}$$

Letting $r = i \cos \phi + j \sin \phi \cos \psi + k \sin \phi \sin \psi$ we have

$$(a,b,c,d) = u(\cos\theta + r\sin\theta) \tag{41}$$

where $u = \sqrt{a^2+b^2+c^2+d^2}$, denotes the amplitude of (a,b,c,d) and r represents the more general second root of negative one as given by equation (40).

Consider now the equation

$(a,b,c,d)^n = u^n(\cos\theta + r\sin\theta)^n$ where n is a positive integer. If $n = 1$, then

$(\cos\theta + r\sin\theta)^n = \cos n\theta + r\sin n\theta$. Assuming

$(\cos\theta + r\sin\theta)^k = \cos k\theta + r\sin k\theta$ for some positive integer k we have that

$$\begin{aligned}
(\cos\theta + r\sin\theta)^{k+1} &= (\cos\theta + r\sin\theta)\ (\cos\theta + r\sin\theta) \\
&= (\cos\theta k + r\sin\theta k)(\cos\theta + r\sin\theta) \\
&= \cos\theta k\cos\theta - \sin\theta k\sin\theta \\
&\quad +r(\cos\theta k\sin\theta + \cos\theta\sin\theta k) \\
&= \cos(\theta k + \theta) + r\sin(\theta k + \theta) \\
&= \cos\theta(k+1) + r\sin\theta(k+1).
\end{aligned}$$

Consequently,

$$(a,b,c,d)^n = u^n(\cos\theta n + r\sin\theta n) \tag{42}$$

for all positive integers n. Equation (42) in turn requires that $(a,b,c,d)^{1/n} = u^{1/n}(\cos\theta\cdot\frac{1}{n} + r\sin\theta\cdot\frac{1}{n})$ for all positive integers n. To establish this fact we consider that

$$(u(\cos\theta + r\sin\theta))^{1/n} = u^{1/n}(\cos\theta + r\sin\theta)^{1/n}.$$

Letting $\theta = n\theta'$, substitution yields

$$(u(\cos\theta + r\sin\theta))^{1/n} = u^{1/n}(\cos n\,\theta' + r\sin n\,\theta')^{1/n}$$
$$= u^{1/n}\,(\cos\theta' + r\sin\theta')^{n\,1/n}$$
$$= u^{1/n}(\cos\theta' + r\sin\theta')$$
$$= u^{1/n}(\cos\frac{\theta}{n} + r\sin\frac{\theta}{n}).$$

Therefore,

$$(a,b,c,d)^{1/n} = u^{1/n}(\cos\frac{\theta}{n} + r\sin\frac{\theta}{n}) \tag{43}$$

for all positive integers n. If m is a non-negative integer we also have that

$$(a,b,c,d)^{1/n} = u^{1/n}\cos(\frac{\theta}{n} + \frac{2\pi m}{n}) + r\sin(\frac{\theta}{n} + \frac{2\pi m}{n}).$$

Notice that, pair wise, $\cos(\frac{\theta}{n} + \frac{2\pi m}{n})$ and $\sin(\frac{\theta}{n} + \frac{2\pi m}{n})$ take on exactly n distinct values. Consequently, $(a,b,c,d)^{1/n}$ has exactly n distinct values given by the equation

$$(a,b,c,d)^{1/n}=u^{1/n}(\cos(\frac{\theta+2\pi m}{n})+ r\sin(\frac{\theta+2\pi m}{n})) \tag{44}$$

where $m = 0, 1, 2, 3, \cdots, n-1$. In particular, the value of $(a,b,c,d)^{1/2}$ can be computed by equation (44) or by the method described below.

Let $(a,b,c,d)^{1/2} = (x,y,z,w)$ then squaring both sides we have $(a,b,c,d) = (x,y,z,w)^2$. By the Law of Moduli

$$\sqrt{a^2+b^2+c^2+d^2} = \sqrt{x^2+y^2+z^2+w^2}\ \sqrt{x^2+y^2+z^2+w^2}.$$

Therefore,

$$\sqrt{a^2+b^2+c^2+d^2} = x^2+y^2+z^2+w^2. \tag{45}$$

If $(a,b,c,d) = (x,y,z,w)$ then

$(a,b,c,d) = (x^2-y^2-z^2-w^2,\ 2xy, 2xz, 2xw)$ and we have

$$a = x^2-y^2-z^2-w^2$$

$$b = 2xy$$

$$c = 2xz$$

$$d = 2xw.$$

Since $2x^2 = (x^2-y^2-z^2-w^2)+(x^2+y^2+z^2+w^2)$ substitution of equation (45) and $a = x^2-y^2-z^2-w^2$ yields $2x = a+\sqrt{a^2+b^2+c^2+d^2}$. Letting $u = \sqrt{a^2+b^2+c^2+d^2}$ we have $x = \frac{\sqrt{2(a+u)}}{2}$. Using this last equation and $b = 2xy$, $c = 2xz$, $d = 2xw$, it follows that $y = b\frac{\sqrt{2(a+u)}}{2(a+u)}$, $z = c\frac{\sqrt{2(a+u)}}{2(a+u)}$, $w = d\frac{\sqrt{2(a+u)}}{2(a+u)}$. Therefore, if $(a,b,c,d) \neq (0,0,0,0)$, then

$$(a,b,c,d)^{1/2} = \left(\frac{\sqrt{2(a+u)}}{2(a+u)},\quad b\frac{\sqrt{2(a+u)}}{2(a+u)},\quad c\frac{\sqrt{2(a+u)}}{2(a+u)},\quad d\frac{\sqrt{2(a+u)}}{2(a+u)}\right)$$

or

$$(a,b,c,d)^{1/2} = \frac{\sqrt{2(a+u)}}{2(a+u)}\,(a+u,b,c,d) \qquad (46)$$

where $u = \sqrt{a^2+b^2+c^2+d^2}$.

Closely related to equation (44) are equations of the form $x^2+px+q=0$ defined over the quaternions. Let $w=(a,b,c,d)$ denote a solution of the equation $x^2+px+q=0$ where p and q are real numbers. By equation (41)

$$\begin{aligned}(a,b,c,d) &= u(\cos\theta + r\sin\theta)\\ &= u\cos\theta + r\,u\sin\theta\\ &= a + r\,m\end{aligned}$$

where $m = u \sin\theta = \sqrt{b^2+c^2+d^2}$ as defined in Chapter IV. Substituting into $x^2+px+q=0$, we obtain $(a+rm)^2+p(a+rm)+q=0$ or $a^2+2mra-m^2+pa+pmr+q=0$. In order for this equation to be satisfied, both the real and imaginary components must have a value of zero. That is,

i) $a^2-m^2+pa+q=0$, and

ii) $2mar+pmr=0$.

From ii), either $m=0$, or $p=-2a$. If $m=0$, then from i), $q=a^2-pa$. If $m\neq 0$, then $p=-2a$ which yields

$$q=a^2+w^2=a^2+b^2+c^2+d^2=u^2$$

where u is the modulus of (a,b,c,d). Thus, we have shown that for (a,b,c,d) to be a solution of $x^2+px+q=0$, it must be the case that i) $q=u^2$, or ii) $q=-a^2-pa$. Given an equation of the form $x^2+px+q=0$, one could use equation (46) in conjunction with the quadratic formula to find solutions.

As a final problem, consider taking the conjugate of (a,b,c,d). Denote the conjugate of (a,b,c,d) by (a´,b´,c´,d´). If such an element is to exist we will have that $(a,b,c,d)(a',b',c',d')=(a',b',c',d')(a,b,c,d)=a^2+b^2+c^2+d^2$. If $(a,b,c,d)(a',b',c',d')=a^2+b^2+c^2+d^2$, then $\frac{1}{u^2}(a,b,c,d)(a',b',c',d')=1$ where $u^2=a^2+b^2+c^2+d^2$. Letting $1=(1,0,0,0)$ we have

$$\left(\frac{a}{u^2}, \frac{b}{u^2}, \frac{c}{u^2}, \frac{d}{u^2}\right)(a',b',c',d') = (1,0,0,0)$$

which indicates (a´,b´,c´,d´) is the multiplicative inverse of $(\frac{a}{u^2}, \frac{b}{u^2}, \frac{c}{u^2}, \frac{d}{u^2})$. From Chapter III, (a,-b,-c,-d) is the inverse of $(\frac{a}{u^2}, \frac{b}{u^2}, \frac{c}{u^2}, \frac{d}{u^2})$ and so

$$(a',b',c',d') = (a,-b,-c,-d).$$

Expressions of the form $\frac{1}{(a,b,c,d)}$ can now be expressed as $\frac{(a,-b,-c,-d)}{a^2+b^2+c^2+d^2}$. Notice that if c=d=0, then the last equation reduces to the conjugate for complex numbers.

CHAPTER VI

SUMMARY

As a consequence of this paper and the forty-six major equations contained herein, we have been able to extend the notion of an imaginary number. One such extension is the set of quaternions.

The system of quaternions was shown to have the same properties as those of the complex number system except for the fact that quaternion multiplication is non-commutative. This discrepancy was illustrated with the use of the imaginary vectors i, j and k.

Although points in space can be described with ordered triples, we have shown that one cannot define a multiplication on triplets consistent with the complex numbers. However, by using quaternions whose first components are zero, one cannot only describe points in space but can also define a binary operation of multiplication which along with quaternion addition determine a skew field.

The algebraic and geometric justification for quaternions was presented in detail and was seen to be consistent with the properties related to the complex numbers.

Although only a small portion of Hamilton's work was presented in this paper, it suffices to point out his genius.

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