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Cauchy Extensions of Integrals

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CHAPTER I

INTRODUCTION

This paper presents some ideas on a different approach to the teaching of integral calculus. The method presented here, though simple in nature, requires little alteration when new topics such as probability and series are developed.

The theory of function rings is developed in Chapter II. Chapter III contains the proof of The Extension Theorem which is the basis of the theory. If "integral" is defined on a function ring, then when the function ring is enlarged to its Cauchy Completion, the integral is extended to the Cauchy Completion.

In the prototype example, "integral" is defined in the evident way for step functions defined on a closed and bounded interval X of the real line. When f is a real-valued function defined on X for which there exists a sequence $\{f_n\}$ of step functions which converges uniformly to f, we define   f to be the limit of the sequence $\{\ f_n\}$. This definition of integral is shown to be legitimate and the characteristics of the function f are examined.

Though algebraic in nature, the proofs do not rely on topological concepts, which is in keeping with the spirit of simplicity. Several theorems yield more general results than necessary but since these extensions require little extra effort, they are stated.

CHAPTER II

FUNCTION RINGS

Let X be a non-empty set and let R^X denote the set of all functions $f: X \to R$ where R denotes the real numbers. The set R^X is both a ring and a lattice under operations defined pointwise; i.e. for $f, g \in R^X$ and all $x \in X$ $(f + g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, $(f \vee g)(x)$ is the larger of $f(x)$ and $g(x)$, and $(f \wedge g)(x)$ is the smaller of $f(x)$ and $g(x)$. Furthermore, $[f \vee (-f)](x) = |f(x)|$ for all $f \in R^X$ and for all $x \in X$ so we use the notation $|f| = f \vee (-f)$. Also the following relations hold in R^X:

2-1) $|f + g| \leq |f| + |g|$

2-2) $-(f \vee g) = (-f) \wedge (-g)$

2-3) $f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h)$

2-4) $f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h)$

2-5) $(f \vee g) + h = (f + h) \vee (g + h)$

2-6) $(f \wedge g) + h = (f + h) \wedge (g + h)$.

Statements 2-3 and 2-4 show the lattice is distributive and statements 2-5 and 2-6 show that in the lattice, additive translations are lattice automorphisms. For each real number r we let r also denote the constant function in R^X defined by $r(x) = r$ for all $x \in X$.

A function ring on X is a pair $(A, \{u_n\})$ where A is a non-empty subring and sub-lattice of R^X and $\{u_n\}$ is a sequence in A such that

2-7) $u_n \geq 0$ but $u_n \neq 0$ for each n in the set N of positive integers,

2-8) if $f \in R^X$ and $0 \leq f \leq u_n$ for all $n \in N$, then $f = 0$,

2-9) $2u_{n+1} = u_{n+1} + u_{n+1} < u_n$ for each $n \in N$,

2-10) if $f \in A$ and $n \in N$, then there exists $m \in N$ such that $|fu_m| \leq u_n$.

It is not required that A have a unity. For an example of a function ring, let X be any non-empty set and let u_n be the constant function $1/2^n$ for each $n \in N$. Let $A = \{f \in R^X \mid$ there exists a constant function c such that $|f| \leq c\}$. Then $(A, \{u_n\})$ is a function ring.

For $(A, \{u_n\})$, a function ring on X, we call two sequences $\{f_n\}$ and $\{g_n\}$ in R^X related if and only if for each $n \in N$, there exists $k \in N$ such that $|f_s - g_t| \leq u_n$ for all $s, t \geq k$; that is, $|f_s - g_t| \leq u_n$ for all s and t sufficiently large. A sequence which is related to itself is called a Cauchy sequence. If related is restricted to the set of all Cauchy sequences in R^X, then related is an equivilance relation. Although related is not an equivilance relation in general, it is transitive and symmetric. A sequence $\{f_n\}$ in R^X is said to have the limit $f \in R^X$

or to converge to $f \in R^X$ if and only if $\{f_n\}$ is related to the sequence $\{g_n\}$ where $g_n = f$ for all $n \in N$. We will write $\{f_n\} \to f$.

THEOREM 1. For $(A, \{u_n\})$ a function ring on X, the limit of a sequence in R^X is unique.

Proof: Suppose $\{f_n\} \to f$, $\{f_n\} \to g$, and $n \in N$. Then $|f_s - f| \leq u_{n+1}$ for all s sufficiently large and $|f_t - g| \leq u_{n+1}$ for all t sufficiently large. Hence $|f - g| \leq |f - f_s| + |f_t - g| \leq u_{n+1} + u_{n+1} \leq u_n$ for all s and t sufficiently large. Thus $|f - g| \leq u_n$ for all $n \in N$ so $|f - g| = 0$.

A function ring $(A, \{u_n\})$ is Cauchy complete if every Cauchy sequence in A has a limit in A.

THEOREM 2. For $(A, \{u_n\})$ a function ring on X, every Cauchy sequence in R^X has a limit in R^X.

Proof: Let $\{g_n\}$ be a Cauchy sequence in R^X. For each $n \in N$ we have $|g_s - g_t| \leq u_n$ for all s and t sufficiently large, hence $|g_s(x) - g_t(x)| \leq u_n(x)$ for all $x \in X$ and s and t sufficiently large. Since $\{u_n(x)\} \to 0$ for all $x \in X$, we see that $\{g_n(x)\}$ is a Cauchy sequence of real numbers. Since R is complete, the sequence $\{g_n(x)\}$ has a limit $g(x) \in R$ for each $x \in X$ thus defining a function $g \in R^X$. We must show for each $n \in N$, there exists $k \in N$ such that $|g_s - g| \leq u_n$ for all $s \geq k$. Suppose $n \in N$ and choose $k \in N$ such that $|g_s - g_t| \leq u_n$ for all $s, t \geq k$. Suppose $x \in X$ and $\varepsilon > 0$, then $|g_t(x) - g(x)| < \varepsilon$ for all t sufficiently large. So if $s, t \geq k$ and t is sufficiently large, we have $|g_s(x) - g(x)| \leq |g_s(x) - g_t(x)| +$ $|g_t(x) - g(x)| \leq u_n(x) + \varepsilon$. Thus for all $\varepsilon > 0$, we have $|g_s(x) - g(x)| \leq u_n(x) + \varepsilon$ whenever $s \geq k$ so $|g_s(x) - g(x)| \leq u_n(x)$ for all $s \geq k$. Note that k was chosen before x and is independent of x so $|g_s - g| \leq u_n$ for all $s \geq k$.

We define a map to be a function $\alpha\colon (A, \{u_n\}) \to (B, \{v_n\})$ from a function ring A on X to a function ring B on Y with the following properties:

2-11) if $f \leq g$ in A, then $\alpha f \leq \alpha g$ in B,

2-12) $\alpha(f + g) = \alpha f + \alpha g$ for all $f, g \in A$,

2-13) for each $n \in N$, there exists $m \in N$ such that if $|f| \leq u_m$, then $|\alpha f| \leq v_n$.

<u>THEOREM 3</u>. Let $\alpha\colon (A, \{u_n\}) \to (B, \{v_n\})$ be a map. If $\{f_n\}$ is a Cauchy sequence in A, then $\{\alpha f_n\}$ is a Cauchy sequence in B.

<u>Proof</u>: Let $n \in N$ and choose $m \in N$ such that $|\alpha f| \leq v_n$ whenever $|f| \leq u_m$. Then for all s and t sufficiently large, $|f_s - f_t| \leq u_m$ and $|\alpha(f_s - f_t)|$ $= |\alpha f_s - \alpha f_t| \leq v_n$ so $\{\alpha f_n\}$ is a Cauchy sequence in B.

CHAPTER III

THE EXTENSION THEOREM

We are now ready to state the Extension Theorem which has been proved in a more general context by J. T. Morse.[1]

<u>THEOREM 4</u>. If $(A, \{u_n\})$ is a function ring on X, then there exists a function ring $(\hat{A}, \{u_n\})$ on X such that $A \subsetneq A$ and each element of $(\hat{A}, \{u_n\})$ is the limit of a sequence in $(A, \{u_n\})$ and $(\hat{A}, \{u_n\})$ is Cauchy complete. If $(B, \{v_n\})$ is a Cauchy complete function ring on Y and α: $(A, \{u_n\}) \to (B, \{v_n\})$ is a map, then α has a unique extension $\hat{\alpha}$: $(\hat{A}, \{u_n\}) \to (B, \{v_n\})$ as a map. In addition, if α preserves multiplication, then so does $\hat{\alpha}$.

<u>Proof</u>. For brevity, we will denote $(A, \{u_n\})$, $(B, \{v_n\})$, and $(\hat{A}, \{u_n\})$ by A, B, and $\hat{A}$ respectively. Define $\hat{A}$ to be the set of all $f \in R^X$ such that there exists a sequence $\{f_n\}$ in A which converges to f. First, we must show $\hat{A}$ is a function ring. Properties 2-7, 2-8, and 2-9 are clearly true so consider property 2-10 of $\{u_n\}$. Let $f \in \hat{A}$ and suppose $n \in N$. Also let $\{f_n\}$ be a sequence in A such that $\{f_n\} \to f$. Choose $m \in N$ such that $u^2 \leq u_{n+1}$ and $m \geq n + 1$. This is possible by choosing

$m \in N$ such that $u_1 u_m \leq u_n$ since $u_m \leq u_1$. Pick $k \in N$ such that $|f - f_k| \leq u_m$ and choose $s \geq m$, $s \in N$, such that $|f_k u_s| \leq u_m$. Then we have $|fu_s| \leq u_s\, |f - f_k| + |f_k u_s|$ $\leq u_s u_m + u_m \leq u_m^2 + u_m \leq u_{n+1} + u_{n+1} \leq u_n$ so 2-10 is satisfied. Now, if $f, g \in \hat{A}$ and if $\{f_n\}$ and $\{g_n\}$ are sequences in A with $\{f_n\} \to f$ and $\{g_n\} \to g$, it is clear that $\{f_n - g_n\} \to f - g \in \hat{A}$. To show $\{f_n g_n\} \to fg$, suppose $n \in N$ and choose $p \in N$ such that $|fu_p| \leq u_{n+1}$. Choose $t \in N$ such that $|u_t(|g| + u_p)| \leq u_{n+1}$. Then for all s sufficiently large, $|f_s g_s - fg| = |f(g_s - g)$ $+(f_s - f)(g_s - g + g)| \leq |fu_p| + |u_t(|g| + u_p)|$ $\leq u_{n+1} + u_{n+1} \leq u_n$ where $|g_s - g| \leq u_p$ and $|f_s - f| \leq u_t$ for all s sufficiently large. For $\hat{A}$ to be a sublattice of R^X, we must show if $f, g \in \hat{A}$, then $f \vee g \in \hat{A}$ and $f \wedge g \in A$. Note that if $\hat{A}$ is closed under $\vee$, it will be closed under $\wedge$ since $f \wedge g = -\left[(-f) \vee (-g)\right]$. Suppose $n \in N$, then $|f_s - f| \leq u_n$ and $|g_s - g| \leq u_n$ for all s sufficiently large where $\{f_n\}$ and $\{g_n\}$ are as above. Furthermore, $f - u_n \leq f_s \leq f + u_n$ and $g - u_n \leq g_s \leq g + u_n$ for all s sufficiently large and $(f \vee g) - u_n \leq f \vee g \leq (f \vee g) + u_n$ or $|(f_s \vee g_s) - (f \vee g)| \leq u_n$ for all s sufficiently large. Thus $\{f_n \vee g_n\} \to f \vee g$ and $f \vee g \in \hat{A}$. $\hat{A}$ is thus seen to be a function ring on X and we must show $\hat{A}$ is Cauchy complete.

Let $\{g_n\}$ be a Cauchy sequence in A. There exists $g \in R^{*}$ such that $\{g_n\} \to g$ by Theorem 2. For each g_n, denote $\{g_{nm}\}$ a sequence in A which has limit g_n. Suppose $s \in N$ and for each $n \in N$, choose $k \in N$ such that $|g_{nk} - g_n| \leq u_{s+1}$ and set $a_n = g_{nk}$. The sequence $\{a_n\}$ is in A and we will show $\{a_n\} \to g$, hence $g \in \hat{A}$. Since $|a_t - g_t| \leq u_{s+1}$ for all t sufficiently large and $|g_t - g| \leq u_{s+1}$ for all t sufficiently large, we have $|a_t - g| \leq |a_t - g_t| + |g_t - g| \leq u_{s+1} + u_{s+1} \leq u_s$ for all t sufficiently large so $\{a_n\} \to g$ and $g \in \hat{A}$. We are now ready to define $\hat{\alpha}$: $\hat{A} \to B$. If $f \in \hat{A}$, there exists a sequence $\{f_n\}$ in A such that $\{f_n\} \to f$. Since $\{f_n\}$ is a Cauchy sequence in A, $\{\alpha f_n\}$ is a Cauchy sequence in B and since B is Cauchy complete, $\{\alpha f_n\}$ has a limit $\hat{f} \in B$. Define $\hat{\alpha} f = \hat{f}$ and we must verify that the choice of $\{f_n\}$ did not effect the selection of $\hat{\alpha} f$. Let $\{g_n\}$ be a different sequence in A with limit f. Suppose $n \in N$, then there exists $m \in N$ such that if $|h| \leq u_m$, then $|\alpha h| \leq v_n$. Since $\{f_n\}$ is related to $\{g_n\}$, there exists $k \in N$ such that for all $s, t \geq k$, $|f_s - g_t| \leq u_m$ which implies $|\alpha f_s - \alpha g_t| \leq v_n$ for all $s, t \geq k$. So $\{\alpha g_n\}$is related to the sequence $\{\alpha f_n\}$ and hence has limit $\hat{f} \in B$ by the uniqueness of $\hat{f}$. Clearly $\hat{\alpha}$ is an extension of α.

For α to be a map, it must satisfy 2-11, 2-12, and 2-13. Consider 2-13 and suppose $n \in N$. Choose $m \in N$

such that if $g \in A$ and $|g| \leq u_{m-1}$, then $|\alpha g| \leq v_{n+1}$. Let $f \in \hat{A}$ with $|f| \leq u_m$ and let $\{f_n\}$ be a sequence in A with limit f. Choose $k \in N$ such that $|f_s - f| \leq u_m$ for all $s \geq k$, then $|f_s| \leq |f_s - f| + |f| \leq u_m + u_m \leq u_{m-1}$ for all $s \geq k$. Hence we have $|\alpha f_s| \leq v_{n+1}$ for all $s \geq k$ so choose $t \in N$ such that $|\hat{\alpha}f - \alpha f_s| \leq v_{n+1}$ for all $s \geq t$. Then for all s sufficiently large, $f \in \hat{A}$ with $|f| \leq u_m$ implies $|\alpha f| \leq |\hat{\alpha}f - \alpha f_s| + |\alpha f_s| \leq v_{n+1} + v_{n+1} \leq v_n$ and 2-13 is satisfied.

Let $f, g \in \hat{A}$ with $\{f_n\}$ and $\{g_n\}$ sequences in A with limits f and g respectively. It is evident that $\{f_n + g_n\} \to f + g$ so by the definition of $\hat{\alpha}$, $\{\alpha(f_n + g_n)\} \to \hat{\alpha}(f + g)$. Since α is a map $\alpha(f_n + g_n) = \alpha f_n + \alpha g_n$ for each $n \in N$, so $\{\alpha f_n + \alpha g_n\} \to \hat{\alpha}f + \hat{\alpha}g$ and by the uniqueness of limits in R^X, $\hat{\alpha}(f + g) = \hat{\alpha}f + \hat{\alpha}g$. Similarly, it is shown that $\hat{\alpha}(fg) = (\hat{\alpha}f)(\hat{\alpha}g)$.

Property 2-11 will now follow if we assume $f, g \in \hat{A}$ as above with $f \leq g$ and note that if $\{f_n\}$ is a sequence in A with limit f, then $\{|f_n|\} \to |f|$. Since $\{g_n - f_n\} \to g - f \geq 0$, $\{|g_n - f_n|\} \to (g - f)$ and by the definition of α, $\{\alpha|g_n - f_n|\} \to \hat{\alpha}(g - f) = \hat{\alpha}g - \hat{\alpha}f$. Since $|g_n - f_n| \geq 0$ for each $n \in N$, $\hat{\alpha}|g_n - f_n| \geq 0$ for each $n \in N$ and hence $\hat{\alpha}g - \hat{\alpha}f \geq 0$ or $\hat{\alpha}f \leq \hat{\alpha}g$.

We conclude our proof by showing $\hat{\alpha}$ is unique. Assume ϕ: $A \to B$ is a map with $\phi f = \hat{\alpha} f$ if $f \in A$. Suppose $f \in \hat{A}$ and $\{f_n\}$ is a sequence in A with limit f. Since $\hat{\alpha}$ and ϕ are both maps, $\{\hat{\alpha} f_n\} \to \hat{\alpha} f$ and $\{\phi f_n\} \to \phi f$. But, $\hat{\alpha} f_n = \phi f_n$ for each $n \in N$, so Theorem 1 implies $\hat{\alpha} f = \phi f$.

CHAPTER IV

INTEGRATION

Consider any non-empty set X and a non-empty subset I of R^X with the following properties:

4-1) The elements of I are idempotents; i.e. $f \in I$ implies $f^2 = f$,

4-2) I is closed under the operations $\vee$ and $\wedge$,

4-3) for each $f \in I$, there exists $g \in I$ such that $f \wedge g = 0$ and $f \vee g = 1$ where 1 is the constant 1-valued function on X,

4-4) there exists a function $\mathcal{I}$: $I \to R$ such that

4-4a) if $f, g \in I$ with $f \leq g$, then $\mathcal{I}(f) \leq \mathcal{I}(g)$,

4-4b) if $f, g \in I$ and $f \wedge g = 0$, then $\mathcal{I}(f \vee g) = \mathcal{I}(f) + \mathcal{I}(g)$.

Remark 4.1. Note that the operations $\wedge$ and multiplication are identical on I. By 4-2 and 4-3, the constant functions 0 and 1 are elements of I. From 4-4a and 4-4b it follows that $\mathcal{I}(0) = 0$ and $\mathcal{I}(f) \geq 0$ for all $f \in I$.

Definition 3.1: If $f \in R^X$ and $f(X)$ is finite, then f is a step function if and only if for each $a \in R$, there exists $g \in I$ such that for all $x \in X$, $f(x) = a$ if and only if $g(x) = 1$. Note that the function $g \in I$ is necessarily unique. Let S be the set of all step functions in R^X. Note that each step function f can be written uniquely in the form $f = a_1h_1 + \cdots + a_jh_j$ where each $h_i \in I$ and $f(x) = a_i$ if and only if $h_i(x) = 1$ for all $x \in X$ and each $i = 1, 2, \cdots, j$. We call $a_1h_1 + a_2h_2 + \cdots + a_jh_j$ the standard decomposition of f.

<u>THEOREM 5</u>. The set S of all step functions is a sublattice and subring of R^X.

<u>Proof</u>. To show S is a sublattice of R^X, suppose f, g $\in$ S and a $\in$ R. There exists h, k $\in$ I such that for all x $\in$ X we have $f(x) = a$ if and only if $h(x) = 1$ and $g(x) = a$ if and only if $k(x) = 1$. Therefore $(f \vee g)(x) = a$ if and only if $h(x) = 1$ or $k(x) = 1$ so $(f \vee g)(x) = a$ if and only if $(h \vee k)(x) = 1$. Since h, k $\in$ I, $h \vee k \in I$ and $f \vee g$ is a step function. We note that $f \wedge g = -\left[(-f) \vee (-g)\right]$ and if f, g $\in$ S, $f \wedge g \in S$ is evident. Suppose f, g $\in$ S and g has the standard decomposition $g = b_1k_1 + b_2k_2 + \quad + b_nk_n$. By evident induction on n, it is sufficient to consider the case n = 1 and $b_1 \neq 0$ to prove S is closed under addition. Suppose a $\in$ R and choose s, t $\in$ I such that $f(x) = a$ if and only if $s(x) = 1$ and $f(x) = a - b_1$ if and only if $t(x) = 1$. Choose h $\in$ I such that $k_1 \wedge h = 0$ and $k_1 \vee h = 1$. Then $(f + g)(x) = a$ if and only if $(s \wedge h) \vee (t \wedge k_1)(x) = 1$; i.e. that is $(f(x) = a$ and $g(x) = 0)$ or $(f(x) = a - b_1$ and $g(x) = b_1)$. But $(s \wedge h) \vee (t \wedge k_1) \in I$, hence $f + g \in S$. Now that

closure under addition is proved, by the distributive laws of R^X it is sufficient in proving closure under multiplication to consider the case where f and g are each constant multiplies of elements of I. But this case is evident since I is closed under $\wedge$, hence closed under multiplication.

Remark 3.2. The set S of all step functions is the smallest subring and sublattice of R^X which contains I and all constant functions.

Let $f \in S$ with standard decomposition $f = a_1f_1 + a_2f_2 + \cdots + a_nf_n$. Define a function $\int: S \to R$ where R denotes the set of real numbers by $\int f = a_1\mathcal{I}(f_1) + a_2\mathcal{I}(f_2) + \cdots + a_n\mathcal{I}(f_n)$. Note that S restricted to the elements of I is equal to $\mathcal{I}$ so we write $\int$ in place of $\mathcal{I}$ for elements of I. Also if f is a step function and $f \geq 0$, then clearly $\int f \geq 0$.

THEOREM 6. The function $\int$ on S has the following properties for all f, $g \in S$:

4-5) $\int af = a\int f$ for all $a \in R$,

4-6) $\int f + g = \int f + \int g$,

4-7) If $f \leq g$, then $\int f \leq \int g$.

Proof: Statement 4-5 is evident from the definition of on S. For 4-6, it is sufficient to consider the case $g = ah$ for $a \in R$, $a \neq 0$, and $h \in I$ as was done in Theorem 5. Let $f = a_1f_1 + a_2f_2 + \cdots + a_nf_n$ be the standard decomposition of f and let $w_i \in I$ such that $f_i \wedge w_i = 0$ and $f_i \vee w_i = 1$

for $i = 1, 2, \cdots, n$ and $w \in I$ such that $h \wedge w = 0$ and $h \vee w = 1$. Then $f + g = (a_1 + a)(f_1 \wedge h) + \cdots + (a_n + a)(f_n \wedge h) + a_1(f_1 \wedge w) + \cdots + a_n(f_n \wedge w) + a(h \wedge w_1 \wedge \cdots \wedge w_n)$ and each distinct pair of summands has lattice $\wedge$ equal to zero. Thus by the required properties of $\int$ on I, $\int(f + g) = (a_1 + a)\int(f_1 \wedge h) + \cdots + (a_n + a)\int(f_n \wedge h) + a_1\int(f_1 \wedge w) + \cdots + a_n\int(f_n \wedge w) + a\int(h \wedge w_1 \wedge \dots \wedge w_n)$. Note that $\int f_i = \int[(f_i \wedge h) \vee (f_i \wedge w)] = \int(f_i \wedge h) + \int f_i \wedge w$ for $i = 1, \cdots, n$, so $\int(f + g) = a_1\int f_1 + a_2\int f_2 + \cdots + a_n\int f_n + a\int(f_1 \wedge h) + \cdots + a\int(f_n \wedge h) + a\int(h \wedge w_1 \wedge \cdots \wedge w_n) = \int f + a\int h = \int f + \int g$ since $\int(f_1 \wedge h) + \cdots + \int(f_n \wedge h) + \int(h \wedge w_1 \wedge \cdots \wedge w_n) = \int[(f_1 \quad \cdots \quad f_n) \wedge h] + \int(h \wedge w_1 \wedge \cdots \wedge w_n) = \int([(f_1 \vee \cdots \vee f_n) \vee (w_1 \vee \cdots \vee w_n)] \wedge h) = \int(1 \wedge h) = \int h$. For the proof of 4-7, note that if $f \leq g$, then $g = f + (g - f)$ and $(g - f) \geq 0$, so apply 4-6 to obtain $\int g = \int f + \int(g - f) \geq \int f$.

The set S is not a function ring as a sequence $\{u \ \}$ in S must be defined with properties 2-7 through 2-10. As soon as this is done, $(S, \{u_n\})$ becomes a function ring. If property 2-13 is verified for $\int$: $S \to R$, then by noting that the real numbers are a Cauchy complete function ring when a proper sequence $\{v_n\}$ is defined, the Extension Theorem may be applied to $(S, \{u_m\})$ and $\int$.

CHAPTER V

EXAMPLES AND APPLICATIONS

Example 1. Let $[a, b]$ be a closed and bounded interval in the set R of real numbers with $a < b$. The set I will consist of all idempotent functions $f \in R^{X}$ for which there exists a partition $T = \{t_0, t_1, \cdots, t_n\}$ of X with $a = t_0 < t_1 < \cdots < t_n = b$ such that if $t_{i-1} < x < t_i$ and $t_{i-1} < y < t_i$, then $f(x) = f(y)$ for each $i = 1, \cdots, n$. In other words, f is constant on each interval (t_{i-1}, t_i) and the value of f at the end points of (t_{i-1}, t_i) need not agree with the value of f on (t_{i-1}, t_i). Define $\mathcal{J}: I \to R$ by $\mathcal{J}(f) = a_1(t_1 - t_0) + a_2(t_2 - t_1) + \cdots + a_n(t_n - t_{n-1})$ for $f \in I$ and $a_i = f\left(\frac{t_i - t_{i-1}}{2}\right)$. The definition of $\mathcal{J}$ is independent of the partition T on $X^{(2)}$. The set I is a Boolean algebra under the operations $\wedge$ and $\vee$ and the function $\mathcal{J}$ satisfies properties 4-4a and 4-4b. We extend I to the set S of all step functions defined on X. If $f \in S$ and $f = a_1 f_1 + \cdots + a_n f_n$ is the standard decomposition of f, then $\int: S \to R$ defined by $\int f = a_1 \mathcal{J}(f_1) + \cdots + a_n \mathcal{J}(f_n)$ is an extension of $\mathcal{J}$ to S. Theorem 6 shows that $\int \alpha f = \alpha \int f$ for any $\alpha \in R$, $f \in S$ and $\int f + g = \int f + \int g$ for all $f, g \in S$. We would like to apply the Extension Theorem to S to obtain a larger class of integrable functions. To make S a function ring, it suffices to exhibit a sequence $\{u_n\}$ in S which satisfies properties 2-7 through 2-10 since S is a subring and sublattice of R^X by Theorem 5. The sequence of constant functions $\{u_n\}$ defined by $u_n(x) = 1/2n$ for all $x \in X$ will clearly suffice.

We note that the definition of the sequence $\{u_n\}$ will restrict the function in the set $(S, \{u_n\})$. If the sequence $\{u_n\}$ is defined as above, then for each $n \in N$, $u_n \neq 0$ so by property 2-10 we see that $|f| \leq u_n/u_m$ or each f in $(S, \{u_n\})$ is bounded. Furthermore, it is evident that if $f \in (S, \{u_n\})$, then f is bounded on the set of all $x \in X$ such that $u_n(x) \neq 0$ for all $n \in N$.

The real numbers R are a ring and a lattice so if we define a sequence $\{v_n\}$ in R by $v_n = 1/2^n$ for each $n \in N$, then $(R, \{v_n\})$ is a function ring. The function $\int: (S, \{u_n\}) \to (R, \{v_n\})$ will be a map if property 2-13 is satisfied since 2-11 and 2-12 are verified in Theorem 5. Suppose $n \in N$. We must show there exists $m \in N$ such that if $|f| \leq u_m$, then $|\int f| \leq v_n = 1/2^n$. Choose $m \in N$ such that $(1/2^m) \int 1 \leq 1/2^n$ where 1 is the constant function $1(x) = 1$ for all $x \in X$. Note that if $f \in S$, $|\int f| \leq \int |f|$ so if $|f| \leq u_m = 1/2^m\, 1$, then $|\int f| \leq \int |f| \leq \int u_m = 1/2^m \int 1 \leq v_n$. Hence property 2-13 is satisfied and $\int: S \to R$ is a map.

The Extension Theorem may now be applied to S and $\int$. Let $f \in R^X$ and $x \in X$. If for each sequence $\{x_n\}$ in X with $\{x_n\} \to x$ and $x_n > x$ for each n, the sequence $\{f(x_n)\}$ converges in the real numbers R, then f is said to have a right-hand limit at x. This is equivilant to saying that $\lim_{0<h\to 0} f(x + h)$ exists. The left-hand limit of f at x is similarly defined.

THEOREM 7. The set S consists of all $f \in R^X$ such that the left-hand and right-hand limits exist at each point $x \in X$.[3]

Proof. Let $f \in S$, $t_0 \in X$, and $\{f_n\}$ a sequence in S with limit f. We will show the right-hand limit exists at t_0 and observe that the left-hand limit is similarly shown. Let $\{t_k\} \to t_0$ with each $t_k > t_0$. If we can show that $\{f(t_k)\}$ is a Cauchy sequence in R, then since R is complete, $\{f(t_k)\}$ converges. Suppose $n \in N$, then there exists $k \in N$ such that $|f_m - f| \leq u_{n+2}$ for all $m \geq k$. Since $f_m \in S$, $\lim_{0<h\to 0} f_m(t_0 + h)$ exists and there is $\delta > 0$ such that if $t_0 < t_s \leq t_q < t_0 + \delta$, then $|f_m(t_s) - f_m(t_q)| \leq 1/2^{n+1}$. Hence if $m \geq k$ and $t_0 < t_s \leq t_q < t_0 + \delta$, then $|f(t_s) - f(t_q)| \leq |f(t_s) - f_m(t_s)| + |f_m(t_s) - f_m(t_q)| + |f_m(t_q) - f(t_q)| \leq 1/2^{n+2} + 1/2^{n+1} + 1/2^{n+2} = 1/2^n = v_n$. Therefore the right-hand limit exists for f at t_0 and we may now focus our attention on the converse.

Let $f \in R^X$ such that the left-hand and right-hand limits of f exist for each $t \in X$. Suppose $n \in N$. For each $t_0 \in X$, there exists a $\delta_{t_0} \in R$, $\delta_{t_0} > 0$, such that if $t_0 < t_1 \leq t_2 < t_0 + \delta_{t_0}$, then $|f(t_1) - f(t_2)| \leq 1/2^n$ and if $t_0 - \delta_{t_0} < t_1^1 \leq t_2^1 < t_0$, then $|f(t_1^1) + f(t_2^1)| \leq 1/2^n$. The class of all intervals $\{t: |t - t_0| < \delta_{t_0}\}$, one for each $t_0 \in X$, will cover X. Since X is a compact set, a finite subclass of these intervals will also cover X. The finite point set consisting of the endpoints and midpoints of these intervals lies in some natural order in X. If it is given by $a = s_1 < s_2 < \cdots < s_p = b$, then we let $J_i = (s_i, s_{i+1})$ and note that if $t, t^1 \in J_i$, then $|f(t) - f(t^1)| \leq 1/2^n$. Choose any $s^1 \in J_i$ for $i = 1, \cdots, p - 1$ and define $f_n(t) = f(s_i^1)$ if $t \in J_i$, $i = 1, \cdots, p - 1$ or $f_n(t) = f(s_i)$ if $t = s_i$, $i = 1, \cdots, p$. It is clear that $f_n \in S$ for each $n \in N$ and if $m \in N$, there exists $k \in N$ such that $|f_s - f| \leq 1/2^m$ for all $s \geq k$. Therefore the sequence $\{f_n\}$ in S has limit f, hence $f \in \hat{S}$.

Example 2:

Consider the set of all $F: R \to R$ such that F is monotone increasing and bounded both above and below, with $F(a - 0) = \lim_{0<h\to 0} F(a - h)$ and $F(a + 0) = \lim_{0<h\to 0} F(a + h)$ both existing for each $a \in R$. For $a, b \in R$ with $a \leq b$, define $\mathcal{L}([a, b]) = F(b) - F(a - 0)$, $\mathcal{L}((a, b]) = F(b) - F(a)$, $\mathcal{L}([a, b)) = F(b - 0) - F(a - 0)$, and $\mathcal{L}((a, b)) = F(b - 0) - F(a)$. If h is the characteristic function of an interval J, define the integral, $\mathcal{I}$, of h to be $\mathcal{L}(J)$. The theory in the thesis applies here. It is used in probability and statistics where $0 \leq F(x) \leq 1$ for all $x \in R$ and F is called a distribution function. In classical analysis, the resulting integral is called the Stieltjes integral.

BIBLIOGRAPHY

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3. Morse, J. T. Cauchy Completions for f-rings, 4.