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Asymptotic Behavior of Fluid Jets

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ASYMPTOTIC BEHAVIOR
OF FLUID JETS

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CHAPTER I.

INTRODUCTION

In this paper we shall consider the steady-state, two dimensional, irrotational flow of an inviscid, incompressible fluid under gravity. The flow boundary contains an analytic "fixed boundary arc" Γ_a and a "free boundary arc" Γ_f. These meet at infinity and form the boundaries of a jet of fluid having a source or sink at infinity, where the velocity becomes infinite.

One application of this is the flow past a gas bubble in an infinitely long vertical tube. (2: 1-5). Another is the flow along an inclined plane. (1: 1-5). In our problem we shall consider the special case where Γ_a is a straight line and will use a method of solution involving primarily differential equations.

We consider the fixed boundary to be analytic at infinity and assume the velocity approaches infinity on the free boundary. At other points in the jet, we assume the flow is free of singularities and stagnation points. The principal result we achieve is the asymptotic expansion for the flow through methods of differential equations; the method being applicable to other problems. It may be seen that the velocity approaches infinity throughout the jet and hence becomes thin and asymptotic to the Γ_a.

We choose a complex coordinate z in the flow plane such that the fixed boundary is asymptotic to the positive real axis at infinity. The inverse velocity potential function $z = f(\omega)$ describes the flow near infinity. This is a conformal mapping of a horizontal strip $\underline{S}$ onto the jet. This means $f(\infty) = \infty$ and the boundaries of $\underline{S}$ are mapped into Γ_f and Γ_a. L_f and L_a are the half-lines in the ω-plane which correspond to Γ_f and Γ_a. The strip width A is the source or sink strength.

Now $u(\omega) = \frac{1}{f'(\omega)}$ represents the conjugate flow velocity. Thus, the condition of constant pressure on the free boundary implies $|f'(\omega)|^{-2} - 2g\mathrm{Re}\{f(\omega)\}$ is constant on L_f where g is the absolute specific force. Since Γ_a is a straight line we have $f(\omega) > 0$ on L_a. The fact that the free boundary velocity approaches infinity implies that $f'(\omega)$ tends to zero near infinity on L_f.

Earlier studies have shown that f and u have infinite asymptotic expansions in terms of $\omega^{-1/3}$ and $\log \omega$. (1:35-50). These were for the flow orientation in which there is a sink at infinity and the flow domain lies on the left side of Γ_a. The corresponding expansions for other "flow orientations" are easily obtained by transformations. This implies that when the singularity at infinity is a sink, we have

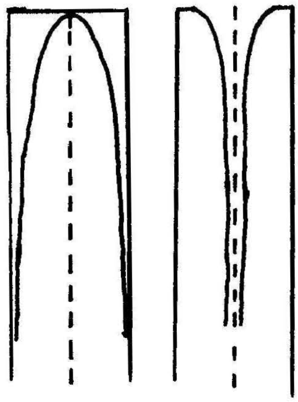
$$f(\omega) \sim f_0 + \left(\frac{9\omega^2}{8g}\right)^{1/3} \left[1+P(\omega)\right] \text{ and } u(\omega) \sim (3g\omega)^{1/3} \left[1+Q(\omega)\right]$$

where f_0 is a real constant and P and Q are infinite series in powers of ω^{-1}. The constant in our differential equation turns out to be $-2gf_0$. (1: 4).

Given g, A, Γ_a, and the "flow orientation", it turns out that f and u are uniquely determined by f_0 and a second constant ω_0 which represents the arbitrary additive constant in the complex velocity potential. It has been shown that changes in the boundary conditions and singularities in the flow can affect the asymptotic jet behavior only through the parameter f_0. Also, the free boundary shape is given by $XY^2 \sim \frac{A^2}{2g}$ and hence is independent of f_0. (1: 5).

CHAPTER II.

THE PROBLEM



One case of the physical problem involves the flow past a gas bubble in an infinitely long vertical tube. We choose a coordinate system attached to the bubble, with the liquid flowing around the bubble. We place the apex of the bubble on the imaginary axis with the fixed boundary on the real axis. Gravity is acting downward in the tube. The flow is described in terms of a complex potential which is an analytic function in the physical plane. The tube is represented in the z-plane by an infinite strip of width A. The condition of constant pressure on the free boundary takes the form $|f'(\omega)|^{-2}-2g\mathrm{Re}\{f(\omega)\}$ is constant on L_f. This is implied by Bernoullis' equation. (2: 1).

Another case is the flow from a slot. The corners of the slot and the apex of the bubble are points of singularity. We consider the standard orientation in which there is a sink at infinity and the flow domain lies on the left side of the fixed boundary. This means that the half strip S extends to $\mathrm{Re}\{\omega\} = \infty$ and L_a lies below L_f.

We map the strip S conformally into the upper and lower semicircle by the functions $\zeta(\omega) = \exp(\frac{-\omega\pi}{A}+i\pi)$ and

$\zeta(\omega) = \exp(\frac{-\omega\pi}{A})$. We map the free boundary onto the real axis and map ∞ onto the origin. Our condition of constant pressure is independent of the shape of the boundary. Now the flow along each side of the bubble and from each corner of the slot will all represent the same physical problem with our vertical boundary.

CHAPTER III.

METHOD OF SOLUTION

We consider the family of all functions $F(\omega)$ such that F is defined and regular on S and real on L_a. We establish asymptotic expansions for f and u from the function $F = \log u$. This implies that $u = e^F$ and so $\frac{1}{u} = e^{-F}$. From the conjugate flow velocity we get $f'(\omega) = e^{-F(\omega)}$. Hence we can write the flow $f(\omega)$ as $\int^{\omega} e^{-F(\omega')}d\omega'$.

We choose the functions v and ϕ such that v is analytic in the strip S, $\phi = \arg u$, and $v = |u|$ on the free boundary. We thus have $F = \log u = \log v + i\phi$. Hence on the boundary the real and imaginary parts of $F(\tau+iA)$ are given by $G = \mathrm{Re}\{F\} = \log v$ and $H = \mathrm{Im}\{F\} = \phi$. Earlier studies give us the expansions for F and u as

(1) $$F(\omega) = \frac{1}{3}\log(3g\omega) + \frac{\omega_0}{3\omega} - \frac{9\omega_0^2+8A^2}{54\omega^2} + \frac{\alpha^3}{\omega^3} + \cdots$$

(2) $$u(\omega) = (3g\omega)^{1/3}\left[1 + \frac{\omega_0}{3\omega} - \frac{3\omega_0^2+4A^2}{27\omega^2} + \frac{\delta^3}{\omega^3} + \cdots\right].$$ (1: 39)

The condition of constant pressure on the free boundary implies $|f'(\omega)|^{-2} - 2g\,\mathrm{Re}\{f(\omega)\}$ is constant on L_f. From the substitutions above we can write $v^2(\omega) - 2g\,\mathrm{Re}\{\int^{\omega}\frac{1}{u(\omega')}\,d\omega'\}$ is constant on L_f. Setting $\omega = \tau+iA$ and $f' = \frac{1}{u} = e^{-i\phi}/v$ we get

$\mathrm{Re}\{\int^{\omega} \frac{1}{u(\omega')}\,d\omega'\} = \mathrm{Re}\{\int_{1+iA}^{\tau+iA} \frac{e^{-i\phi(\omega')}}{v(\omega')}\,d\omega'\}$. Hence

$\mathrm{Re}\{\int^{\omega} \frac{1}{u(\omega')}\,d\omega'\} = \mathrm{Re}\{\int_{1}^{\tau} \frac{e^{-i\phi(x+iA)}}{v(x+iA)}\,dx\}$. Thus we have

$v^2(\omega) - 2g\mathrm{Re}\{\int_{1}^{\tau} \frac{\cos\phi(x+iA)}{v(x+iA)}\,dx\}$ is constant. Differentiate

and we get $2v(\omega)\frac{dv}{d\omega} - 2g\frac{\cos\phi(\omega)}{v(\omega)} = 0$. Hence $\frac{dv}{d\omega} = \frac{g\cos\phi(\omega)}{v^2(\omega)}$

on L_f. Now $\cos\phi = \frac{u}{v} + \frac{v}{u}$ so we get the differential equation

$\frac{dv}{d\omega} = \frac{g}{v^2}\left(\frac{u}{v} + \frac{v}{u}\right)$.

Let $\omega = \frac{A}{\pi}(-z^{-1} + i\pi)$, $c = \left(\frac{3Ag}{\pi}\right)^{1/3}$, $\hat{u} = uz^{1/3}$,

$\hat{v} = vz^{1/3}$, $\eta = \hat{u} + c$ and $\mu = \hat{v} + c$. μ will be real since we map the free boundary onto the real axis. These substitutions will yield

$$(3)\qquad z\frac{d\mu}{dz} = \frac{\mu - c}{3} + \frac{c^3}{6}\left[(\eta - c)(-c+\mu)^{-3} + (-c+\eta)^{-1}(-c+\mu)^{-1}\right].$$

Evaluate $u\left[\frac{A}{\pi}(-z^{-1}+i\pi)\right]$ to order $\frac{5}{3}$. This will enable us to find η to order 2. Expand η to get η^2 and η^3 to order 3. Expanding (3) to order 3 and substituting for the powers of η we get

$$(4)\qquad z\frac{d\mu}{dz} - \mu = \sum_{\substack{n=0\\ m=0}}^{n+m=3} a_\alpha \mu^n z^m.$$

Using the approximation $\mu = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ where the c_i are necessarily real we have

$\frac{d\mu}{dz} = c_1 + 2c_2z + 3c_3z^2 + \cdots$. Hence we find that $z\frac{d\mu}{dz} - \mu = c_2z^2 + 2c_3z^3 + \cdots$. Comparing this with the coefficients in (4) and using the fact that the c_i are real we can find c_1. Having found c_1 we can then find c_2 and c_3. This will give us $\mu(z)$ to order 3.

Now $\hat{v}(z) = \mu(z) - c$, $v = \hat{v}z^{-1/3}$, and $z = \frac{-A}{\pi\tau}$ so we can evaluate $v(\tau)$ to order $\frac{-8}{3}$. Taking the log v we get an expansion for Re{F} to order -3. Expand (1) at $\tau + iA$ to order -3 and we get another expansion for Re{F}. Comparing coefficients we can find α_3. Next expand $e^{F(\omega)}$ and we have δ_3. This completes our cycle. We start again to the next highest order to find c_4, α_4, and δ_4. This formal process is demonstrated in Chapter 4.

CHAPTER IV.

CALCULATIONS

We have the differential equation $\frac{dv}{d\omega} = \frac{g}{2v^2}\left(\frac{u}{v} + \frac{v}{u}\right)$.

Let $\omega = \frac{A}{\pi}(-z^{-1}+\pi i)$ so $\frac{d\omega}{dz} = \frac{A}{\pi} z^{-2}$ and $z = \frac{A}{\pi}(Ai - \omega)^{-1}$.

By the chain rule we get $\frac{dv}{dz} = \frac{A}{\pi} z^{-2} \frac{dv}{d\omega}$. Hence

$\frac{dv}{dz} \quad \frac{Agz^{\;2}}{2\pi v^2}\left(\frac{u}{v} + \frac{v}{u}\right)$. Let $\hat{v} = vz^{1/3}$ and $\hat{u} = uz^{1/3}$ so

$\frac{d\hat{v}}{dz} = \frac{v}{3} z^{-2/3} + z^{1/3}\frac{dv}{dz}$. Thus $z\frac{d\hat{v}}{dz} = \frac{\hat{v}}{3} + \frac{3Ag}{6\Pi v^2}\left(\frac{u}{v} + \frac{v}{u}\right)$. Let

$c = \left(\frac{3Ag}{\pi}\right)^{1/3}$, $\eta = \hat{u} + c$, and $\mu = \hat{v} + c$. Then $\frac{dv}{dz} = \frac{d\mu}{dz}$ and

so $z\frac{d\mu}{dz} = \frac{\mu-c}{3} + \frac{c^3}{6}(\mu-c)^{-2}\left(\frac{\eta-c}{\mu-c} + \frac{\mu-c}{\eta-c}\right)$. Hence

$$z\frac{d\mu}{dz} = \frac{\mu-c}{3} + \frac{c^3}{6}\quad(\eta-c)(-c+\mu)^{-3} + (-c+\eta)^{-1}(-c+\mu)^{-1}\ .$$

Expanding by the binomial theorem we get

$$z\frac{d\mu}{dz} = \frac{\mu-c}{3} + \frac{c^3}{6}\Big[(\eta-c)(-c^{-3}-3c^{-4}\mu-6c^{-5}\mu^2-10c^{-6}\mu^3) +$$

$$(-c^{-1}-c^{-2}\eta-c^{-3}\eta^2-c^{-4}\eta^3)(-c^{-1}-c^{-2}\mu-c^{-3}\mu^2-c^{-4}\mu^3)\Big] + o(z^4).$$

$$z\frac{d\mu}{dz} = \frac{\mu-c}{3} + \frac{1}{6c^2}\Big[-c^2\eta-3c\eta\mu-6\eta\mu^2+c^3+3c^2\mu+6c\mu^2+$$

$$10\mu^3+c^3+c^2\mu+c\mu^2+\mu^3+c^2\eta+c\eta\mu+n\mu^2+c\eta^2+\eta^2\mu+\eta^3\Big] + o(z^4).$$

Thus $z\frac{d\mu}{dz} - \mu = \frac{1}{6c^2}\Big[-2c\eta\mu-5\eta\mu^2+7c\mu^2+11\mu^3+c\eta^2+\eta^2\mu+\eta^3\Big] + o(z^4)$.

Now $u(\omega) = (3g\omega)^{1/3}\left[1 + \frac{\omega_0}{3\omega} - \frac{3\omega_0^2+4A^2}{27\omega^2} + o(\omega^{-3})\right]$. Hence

$$u(z) = c(-z^{-1}+\pi i)^{1/3}\left[1 + \frac{\omega_0\pi}{3A(-z^{-1}+\pi i)} \quad \frac{(3\omega^2+4A^2)\pi^2}{27A^2(-z^{-1}+\pi i)^2}\right].$$

$$u(z) = cz^{-1/3}\left[-1 + \frac{\pi}{3A}(\omega_0+iA) + \frac{\pi^2}{27A^2}(A^2+6\omega_0 iA+3\omega_0^2)z^2+o(z^3)\right].$$

$\eta = uz^{1/3} + c$ so $\eta = \frac{c\pi}{3A}(\omega_0+iA)+\frac{c\pi^2}{27A^2}(A^2+6\omega_0 iA+3\omega_0^2)z^2+o(z^3)$.

$\eta^2 = \frac{c^2\pi^2}{9A^2}(\omega_0^2+2\omega_0 iA-A^2)z^2+\frac{2c^2\pi^3}{81A^3}(iA^3-5\omega_0 A^2+9\omega_0^2 iA+3\omega_0^3)z^3 +$ $o(z^4)$, and $\eta^3 = \frac{c^3\pi^3}{27A^3}(\omega_0^3+3\omega_0^2 iA-3\omega_0 A^2-iA^3)z^3 + o(z^4)$. Thus

$$z\frac{d\mu}{dz} - \mu = \frac{1}{6c^2}\left[\frac{-2c^2\pi}{3A}(\omega_0+iA)\mu z-\frac{2c^2\pi^2}{27A^2}(A^2+6\omega_0 iA+3\omega_0^2)\mu z^2 + \frac{-5c\pi}{3A}(\omega_0+iA)\mu^2 z+7c\mu^2+11\mu^3+\frac{c^3\pi^2}{9A^2}(\omega_0^2+2\omega_0 iA-A^2)z^2 + \frac{2c^3\pi^3}{81A^3}(iA^3-5\omega_0 A^2+9\omega_0^2 iA+3\omega_0^3)z^3+\frac{c^2\pi^2}{9A^2}(\omega_0^2+2\omega_0 iA-A^2)\mu z^2 +\frac{c^3\pi^3}{27A^3}(\omega_0^3+3\omega_0^2 iA-3\omega_0 A^2-iA^3)z^3\right] + o(z^4).$$

Our first approximation for μ is $c_1 z + c_2 z^2 + c_3 z^3$ where the c_i are real because μ is real on the boundary. From this $z\frac{d\mu}{dz} - \mu = c_2 z^2 + 2c_3 z^3$. Comparing this with the above we get that

$$c_2 = \frac{1}{6c^2}\left[\frac{-2c^2\pi}{3A}(\omega_0+iA)c_1 + 7cc_1^2 + \frac{c^3\pi^2}{9A^2}(\omega_0^2+2\omega_0 iA-A^2)\right].$$

$$\mathrm{Im}\{c_2\} = \frac{1}{6c^2}\left[\frac{-2c^2\pi}{3}\,c_1 + \frac{2c^3\pi^2\omega_0}{9A}\right] = 0. \quad \text{Hence } c_1 = \underline{\underline{\frac{c\omega_0\pi}{3A}}}\,.$$

$$\text{Thus } c_2 = \frac{1}{6c^2}\left[\frac{-2c^2\pi\omega_0}{3A}\left(\frac{c\omega_0\pi}{3A}\right) + \frac{7c^3\omega_0^2\pi^2}{9A^2} + \frac{c^3\pi^2}{9A^2}\left(\omega_0^2 - A^2\right)\right].$$

$$\text{Hence } c_2 = \frac{c\pi^2}{54A^2}\left[-2\omega_0^2 + 7\omega_0^2 + \omega_0^2 - A^2\right] = \underline{\underline{\frac{c\pi^2}{54A^2}\left(6\omega_0^2 - A^2\right)}}$$

$$\text{Now } c_3 = \frac{1}{12c^2}\left[\frac{-2c^2\pi}{3A}\left(\omega_0 + iA\right)\frac{c\pi^2}{54A^2}\left(6\omega_0^2 - A^2\right) - \frac{2c^2\pi^2}{27A^2}\right.$$

$$\left(A^2 + 6\omega_0 iA + 3\omega_0^2\right)\frac{c\omega_0\pi}{3A} - \frac{5c\pi}{3A}\left(\omega_0 + iA\right)\frac{c^2\omega_0^2\pi^2}{9A^2} + \frac{7c^3\omega_0\pi^3}{81A}\left(6\omega_0^2 - A^2\right)$$

$$+ \frac{11c^3\omega_0^3\pi^3}{27A^3} + \frac{2c^3\pi^3}{81A^3}\left(iA^3 - 5\omega_0A^2 + 9\omega_0 iA + 3\omega_0^3\right) + \frac{c^3\omega_0\pi^3}{27A}\,.$$

$$\left.\left(\omega^2 + 2\omega_0 iA - A^2\right) + \frac{c^3\pi^3}{27A^3}\left(\omega_0^3 + 3\omega_0^2 iA - 3\omega_0A^2 - iA^3\right)\right].$$

$$\mathrm{Im}\{c_3\} = \frac{1}{12c^2}\left[\frac{-c^3\pi^3}{81A^2}\left(6\omega_0^2 - A^2\right) - \frac{4c^3\omega_0^2\pi^3}{27A^2} \quad \frac{5c^3\omega_0^2\pi^3}{27A^2}\right.$$

$$\left. + \frac{2c^3\pi^3}{81} + \frac{2c^3\omega_0^2\pi^3}{9A^2} + \frac{2c^3\omega_0^2\pi^3}{27A^2} + \frac{c^3\omega_0^2\pi^3}{9A^2} - \frac{c^3\pi^3}{27}\right]. \quad \mathrm{Im}\{c_3\} =$$

$$\frac{c^3\pi^3}{12c^2}\left[\frac{-2\omega_0^2}{27A^2} + \frac{1}{81} - \frac{4\omega_0^2}{27A^2} - \frac{5\omega_0^2}{27A^2} + \frac{2}{81} + \frac{2\omega_0^2}{9A^2} + \frac{2\omega_0^2}{27A^2} + \frac{\omega_0^2}{9A^2} - \frac{1}{27}\right] = 0.$$

Thus $\mathrm{Im}\{c_3\}$ does indeed equal zero.

$$c_3 = \frac{1}{12c^2}\left[\frac{-c^3\omega_0\pi^3}{81A^3}\left(6\omega_0^2 - A^2\right) - \frac{2c^3\pi^3}{81A^3}\left(A^2 + 3\omega_0^2\right) + \right.$$

$$\frac{-5c^3\omega_0^3\pi^3}{27A^3} + \frac{7c^3\omega_0\pi^3}{81A^3}(6\omega_0^2-A^2) + \frac{11c^3\omega_0^3\pi^3}{27A^3} + \frac{2c^3\pi^3}{81A^3}(-5\omega_0 A^2+3\omega_0^3)$$

$$+ \frac{c^3\omega_0\pi^3}{27A^3}(\omega_0^2-A^2) + \frac{c^3\pi^3}{27A^3}(\omega_0^3-3\omega_0 A^2)\Big]. \quad c_3 = \frac{c\pi^3\omega_0}{12A^3}.$$

$$\Big[\frac{-2\omega_0^2}{27} + \frac{A^2}{81} - \frac{2A^2}{81} - \frac{2\omega_0^2}{27} - \frac{5\omega_0^2}{27} + \frac{14\omega_0^2}{27} - \frac{7A^2}{81} + \frac{11\omega_0^2}{27} - \frac{10A^2}{81} + \frac{2\omega_0^2}{27} +$$

$$\frac{\omega_0^2}{27} - \frac{A^2}{27} + \frac{\omega_0^2}{27} - \frac{A^2}{9}\Big], \quad c_3 = \frac{c\pi^3\omega_0}{12A}\left[\frac{20\omega_0^2}{27} - \frac{10A^2}{27}\right] = \underline{\underline{\frac{5c\pi^3\omega_0}{162A^3}(2\omega_0^2-A^2)}}.$$

At this point $\mu(z) = c\Big[\frac{\omega_0\pi}{3A} z + \frac{\pi^2}{54A^2}(6\omega^2-A^2)z^2$

$+ \frac{5\omega_0\pi^3}{162A^3}(2\omega_0^2-A^2)z^3\Big] + o(z^4)$ and $\hat{v}(z) = c\Big[-1 + \frac{\omega_0\pi}{3A} z +$

$\frac{\pi^2}{54A^2}(6\omega_0^2-A^2)z^2 + \frac{5\omega_0\pi^3}{162A^3}(2\omega_0^2-A^2)z^3\Big] + o(z^4)$. Now

$z = \frac{-A}{\pi\tau}$, $v = \hat{v}z^{-1/3}$, and $c = \left(\frac{3Ag}{\pi}\right)^{1/3}$ so we get that

$$v(\tau) = (3g)^{1/3}\Big[\tau^{1/3} + \frac{\omega_0}{3}\tau^{-2/3} - \frac{1}{54}(6\omega_0^2-A^2)\tau^{-5/3} +$$

$$\frac{5\omega_0}{162}(2\omega_0^2-A^2)\tau^{-8/3} + o(\tau^{-11/3})\Big]. \quad F(\tau+iA) = \frac{1}{3}\log 3g\tau +$$

$$\frac{1}{6}\log\left(1+\frac{A^2}{\tau^2}\right) + \frac{i}{3}\tan^{-1}\frac{A}{\tau} + \frac{\omega_0}{3(\tau+iA)} - \frac{9\omega_0^2+8A^2}{54(\tau+iA)^2} + \frac{\alpha_3}{(\tau+iA)^3} + o(\tau^{-4}).$$

By taking $\frac{A^2}{\tau^2} < 1$ we have $\log(1+\frac{A^2}{\tau^2}) = \frac{A^2}{\tau^2} - \frac{A^4}{2\tau^4} + \frac{A^6}{3\tau^6} + \cdots$

and $\tan^{-1}\frac{A}{\tau} = \frac{A}{\tau} - \frac{A^3}{3\tau^3} + \frac{A^5}{5\tau^5} \cdots$. Hence we get

$$F(\tau+iA) = \frac{1}{3}\log 3g\tau + \frac{\omega_0+iA}{3\tau} \quad \frac{A^2-9\omega_0^2-18\omega_0 iA}{54\tau^2}$$

$$+ \frac{5iA^3+9\omega_0^2 iA-9A^2\omega_0+27\alpha_3}{27\tau^3} + o(\tau^{-4}).$$

$$\operatorname{Log} v = \frac{1}{3}\log 3g\tau + \log\left[1 + \frac{\omega_0}{3\tau} - \frac{6\omega_0^2-A^2}{54\tau^2} + \frac{10\omega_0^3-5\omega_0 A^2}{162\tau^3}\right]$$

so $\log v = \frac{1}{3}\log 3g\tau + \frac{\omega_0}{3\tau} + \frac{A^2-9\omega_0^2}{54\tau^2} + \frac{3\omega_0^3-\omega_0 A^2}{27\tau^3} + o(\tau^{-4})$.

Now $\operatorname{Re}\{F\} = \log v$. Hence $3\omega_0^3-\omega_0 A^2 = -9A^2\omega_0+27\alpha_3$. Thus

$\alpha_3 = \frac{3\omega_0^3+8\omega_0 A^2}{27}$. $\operatorname{Im}\{F\} = \phi$ so $\phi = \frac{A}{3\tau} - \frac{\omega_0 A}{3\tau^2} + \frac{5A^3+9\omega_0^2 A}{27\tau^3}$

$+ o(\tau^{-4})$. $F(\omega) = \frac{1}{3}\log 3g\omega + \frac{\omega_0}{3\omega} - \frac{8A^2+9\omega_0^2}{54\omega^2} + \frac{3\omega_0^3+8\omega_0 A^2}{27\omega^3}$

$+ \frac{\alpha_4}{\omega_4} + o(\omega^{-5})$. $u(\omega) = e^{F(\omega)} = (3g\omega)^{1/3}\left[1 + \frac{\omega_0}{3\omega} - \frac{3\omega_0^2+4A^2}{27\omega^2}\right.$

$\left.+ \frac{1}{81\omega^3}(5\omega_0^3+20\omega_0 A^2) + \frac{123\omega_0^4+336\omega^2{}_0 A^2+32A^4+2916\alpha_4}{2916\omega^4}\right]$.

$$\text{Now } z\frac{d\mu}{dz} = \frac{\mu-c}{3} + \frac{c^3}{6}\left[(\eta-c)(-c^{-3}-3c^{-4}\mu-6c^{-5}\mu^2+\right.$$

$-10c^{-6}\mu^3-15c^{-7}\mu^4) + (c^{-1}-c^{-2}\eta-c^{-3}\eta^2-c^{-4}\eta^3-c^{-5}\eta^4)$

$(-c^{-1}-c^{-2}\mu-c^{-3}\mu^2-c^{-4}\mu^3-c^{-5}\mu^4)\Big]$. $\quad z\,\frac{d\mu}{dz} = \frac{\mu-c}{3} + \frac{c^3}{6}$.

$\Big[-c^{-3}\eta-3c^{-4}\eta\mu-6c^{-5}\eta\mu^2-10c^{-6}\eta\mu^3+c^{-2}+3c^{-3}\mu+6c^{-4}\mu^2+10c^{-5}\mu^3$

$+15c^{-6}\mu^4+c^{-2}+c^{-3}\mu+c^{-4}\mu^2+c^{-5}\mu^3+c^{-6}\mu^4+c^{-3}\eta+c^{-4}\eta\mu+c^{-5}\eta\mu^2$

$+c^{-6}\eta\mu^3+c^{-4}\eta^2+c^{-5}\eta^2\mu+c^{-6}\eta^2\mu^2+c^{-5}\eta^3+c^{-6}\eta^3\mu+c^{-6}\eta^4\Big]$. Hence we get

$$z\,\frac{d\mu}{dz} - \mu = \frac{1}{6c^3}\Big[-2c^2\eta\mu-5c\eta\mu^2-9\eta\mu^3+7c^2\mu^2+11c\mu^3+16\mu^4+$$

$$c^2\eta^2+c\eta^2\mu+\eta^2\mu^2+c\eta^3+\eta^3\mu+\eta^4\Big].$$

$$u(z) = c(-z^{-1}+\pi i)^{1/3}\Big[1 + \frac{\omega_0\pi}{3A(-z^{-1}+\pi i)} - \frac{(3\omega^2+4A^2)\pi^2}{27A^2(-z^{-1}+\pi i)^2}$$

$$+ \frac{(5\omega_0^3+20\omega_0A^2)\pi^3}{81A^3(-z^{-1}+\pi i)^3} + o(z^4)\Big].$$

$$u(z) = c(-z^{-1/3} + \frac{\pi i}{3}\,z^{2/3} - \frac{\pi^2}{9}\,z^{5/3} - \frac{5\pi^3 i}{81}\,z^{8/3})$$

$$\Big[1 + \frac{\omega}{3A}\,(-z-\pi iz^2+\pi^2z^3) - \frac{(3\omega^2+4A^2)\pi^2}{27A^2}\,(z^2+2\pi iz^3) -$$

$$\frac{(5\omega^3+20\omega\ A^2)\pi^2}{81A^3}\,z^3 + o(z^4)\Big].$$

$$u(z) = cz^{-1/3}\Big[-1 + \frac{\pi}{3A}(\omega_0+iA)z + \frac{\pi^2}{27A^2}\,(A^2+6\omega_0iA+3\omega_0^2)z^2$$

$$+ \frac{5\pi^3}{81A^3}\cdot(3iA^3+\omega_0A^2+3\omega_0^2iA+\omega_0^3)z^3 + o(z^4)\Big].$$

Now $\eta = uz^{1/3} + c$ so $\eta = \frac{c\pi}{3A}(\omega_0+iA)z + \frac{c\pi^2}{27A^2}$

$(A^2+6\omega_0 iA+3\omega_0)z^2 + \frac{5c\pi^3}{81A^3}(3iA^3+\omega_0A^2+3\omega_0^2 iA+\omega_0^3)z^3 + o(z^4)$,

$\eta^2 = \frac{c^2\pi^2}{9A^2}(\omega_0^2+2\omega_0 iA-A^2)z^2 + \frac{2c^2\pi^3}{81A^3}(iA^3-5\omega_0A^2+9\omega_0^2 iA+3\omega_0^3)z^3$

$+ \frac{c^2\pi^4}{729A^4}(-89A^4+132\omega_0 iA^3-90\omega_0^2A^2+156\omega_0^3 iA+39\omega_0^4)z^4 + o(z^5)$,

$\eta^3 = \frac{c^3\pi^3}{27A^3}(\omega_0^3+3\omega_0^2 iA-3\omega_0A^2-iA^3)z^3 + \frac{c^3\pi^4}{81A^4}(3\omega_0^4+12\omega_0^3 iA+$

$-14\omega_0^2A^2-4\omega_0 iA^3-A^4)z^4 + o(z^5)$, and $\eta^4 = \frac{c^4\pi^4}{81A^4}(\omega_0^4+4\omega_0^3 iA-$

$6\omega_0^2A^2-4\omega_0 iA^3+A^4)z^4 + o(z^5)$.

Thus $z\frac{d\mu}{dz} - \mu = \frac{1}{6c^3}\Big[-\frac{2c^3\pi}{3A}(\omega_0+iA)\mu z - \frac{2c^3\pi^2}{27A^2}$

$(A^2+6\omega_0 iA+3\omega_0^2)\mu z^2 - \frac{10c^3\pi^3}{81A^3}(3iA^3+\omega_0A^2+3\omega_0^2 iA+\omega_0^3)\mu z^3 - \frac{5c^2\pi}{3A}\cdot$

$(\omega_0+iA)\mu^2 z - \frac{5c^2\pi^2}{27A^2}(A^2+6\omega_0 iA+\omega_0^2)\mu^2 z^2 - \frac{3c\pi}{A}(\omega_0+iA)\mu^3 z$

$+7c^2\mu^2+11c\mu^3+16\mu^4 + \frac{c^4\pi^2}{9A^2}(\omega_0^2+2\omega_0 iA-A^2)z^2 + \frac{2c^4\pi^3}{81A^3}\cdot$

$(iA^3-5\omega_0A^2+9\omega_0^2 iA+3\omega_0^3)z^3 + \frac{c^4\pi^4}{729A^4}(-89A^4+132\omega_0 iA^3-90\omega_0^2A^2$

$+156\omega_0^3 iA+39\omega_0^4)z^4 + \frac{c^3\pi^2}{9A^2}(\omega_0^2+2\omega_0 iA-A^2)\mu z^2 + \frac{2c^3\pi^3}{81A^3}\cdot$

$(iA^3-5\omega_0A^2+9\omega_0^2 iA+3\omega_0^3)\mu z^3 + \frac{c^2\pi^2}{9A^2}(\omega_0^2+2\omega_0 iA-A^2)\mu^2 z^2$

$$+ \frac{c^4\pi^3}{27A^3}(\omega_0^3+3\omega_0^2 iA-3\omega_0 A^2-iA^3)z^3 + \frac{c^4\pi^4}{81A^4}(-A^4-4\omega_0 iA-14\omega_0^2 A^2$$

$$+ 12\omega_0^3 iA+3\omega_0^4)z^4 + \frac{c^3\pi^3}{27A^3}(\omega_0^3+3\omega_0^2 iA-3\omega_0 A^2-iA^3)\mu z^3 + \frac{c^4\pi^4}{81A^4}$$

$$(\omega_0^4+4\omega_0^3 iA-6\omega_0^2A^2-4\omega_0 iA^3+A^4)z^4\Big].$$

Now $\mu = \frac{c\omega_0\pi}{3A} z + \frac{c\pi^2}{54A^2}(6\omega_0^2-A^2)z^2 + \frac{5c\omega_0\pi^3}{162A^3}(2\omega_0^2-A^2)$.

$z^3+c_4z^4$ so we get $z\frac{d\mu}{dz} - \mu = \frac{c\pi^2}{54A^2}(6\omega_0^2-A^2)z^2 + \frac{5c\omega_0\pi^3}{81A^3}$

$(2\omega_0^2-A^2)z^3 + 3c_4z^4$. Comparing this with the above we get

that $$c_4 = \frac{c\pi^4}{18}\Big[-\frac{5\omega_0}{243A^4}(\omega_0+iA)(2\omega_0^2-A^2) - \frac{1}{729A^4}(A^2+6\omega_0 iA+\omega_0^2)$$

$$(6\omega_0^2-A^2) - \frac{10\omega_0}{243A^4}(3iA^3+\omega_0A^2+3\omega_0^2 iA+3\omega_0^3) - \frac{5\omega_0}{243A^4}(\omega_0+iA)$$

$$(6\omega_0^2-A^2) - \frac{5\omega_0^2}{243A^4}(A^2+6\omega_0 iA+3\omega_0^2) - \frac{\omega_0^3}{9A^4}(\omega_0+iA) + \frac{7}{2916A^4}$$

$$(36\omega_0^4-12\omega_0^2A^2+A^4) + \frac{35\omega_0^2}{243A^4}(2\omega_0^2-A^2) + \frac{11\omega_0^2}{162A^4}(6\omega_0^2-A^2)$$

$$+ \frac{16\omega_0^4}{81A^4} + \frac{1}{729A^4}(-89A^4+132\omega_0 iA^3-90\omega_0^2A^2+156\omega_0^3 iA+39\omega_0^4)$$

$$+ \frac{1}{486A^4}(\omega_0^2+2\omega_0 iA-A^2).$$

$$(6\omega_0^2-A^2) + \frac{2\omega_0}{243A^4}(iA^3-5\omega_0 A^2+9\omega_0^2 iA+3\omega_0^3) + \frac{\omega_0^2}{81A^4}(\omega_0^2+2\omega_0 iA-A^2)$$

$$+ \frac{1}{81A^4}(-A^4-4\omega_0 iA^3-14\omega_0^2 A^2+12\omega_0^3 iA+3\omega_0^4) + \frac{\omega_0}{81A}(\omega_0^3+3\omega_0^2 iA$$

$$-3\omega_0 A^2-iA^3) + \frac{1}{81A^4}(\omega_0^4+4\omega_0^3 iA-6\omega_0^2 A^2-4\omega_0 iA^3+A^4)\Big].$$

Now checking to see if $Im\{c_4\} = 0$ we have

$$Im\{c_4\} = \frac{c\pi^4}{18}\Big[-\frac{10\omega_0^3}{243A^3} + \frac{5\omega_0}{243A} - \frac{5\omega_0^3}{81A^3} + \frac{2\omega_0}{243A} - \frac{10\omega_0}{81A} - \frac{10\omega_0^3}{81A^3}$$

$$- \frac{10\omega_0^3}{81A^3} + \frac{5\omega_0}{243A} - \frac{10\omega_0^3}{81A^3} - \frac{\omega_0^3}{9A^3} + \frac{44\omega_0}{243A} + \frac{52\omega_0^3}{243A^3} + \frac{2\omega_0^3}{81A^3} - \frac{\omega_0}{243A}$$

$$+ \frac{2\omega_0^3}{27A} + \frac{2\omega_0^3}{81A} - \frac{4\omega_0}{81A} + \frac{4\omega_0^3}{27A} + \frac{\omega_0^3}{27A} - \frac{\omega_0}{81A} + \frac{4\omega_0^3}{81A} - \frac{4\omega_0}{81A}\Big] = 0.$$

Thus $Im\{c_4\}$ does indeed equal zero.

$$c_4 = \frac{c\pi^4}{18}\Big[-\frac{10\omega_0^4}{243A^4} + \frac{5\omega_0^2}{243A^2} - \frac{2\omega_0^2}{243A^2} + \frac{1}{729} - \frac{2\omega_0^4}{81A^4}$$

$$+ \frac{\omega_0^2}{243A^2} - \frac{10\omega_0^2}{243A^2} - \frac{10\omega_0^4}{243A^4} - \frac{10\omega_0^4}{81A^4} + \frac{5\omega_0^2}{243A^2} - \frac{5\omega_0^2}{243A^2} - \frac{5\omega_0^4}{81A^4}$$

$$- \frac{\omega_0^4}{9A^4} + \frac{7\omega_0^4}{81A^4} - \frac{7\omega_0^2}{243A^2} + \frac{7}{2916} + \frac{70\omega_0^4}{243A^4} - \frac{35\omega_0^2}{243A^2} + \frac{11\omega_0^4}{162A^2}$$

$$+ \frac{16\omega_0^4}{81A^4} - \frac{89}{729} - \frac{10\omega_0^2}{81A^2} + \frac{13\omega_0^4}{243A^4} + \frac{\omega_0^4}{81A^4} - \frac{\omega_0^2}{486A^2} - \frac{\omega_0^2}{81A^2} + \frac{1}{486}$$

$$- \frac{10\omega_0^2}{243A^2} + \frac{2\omega_0^4}{81A^4} + \frac{\omega_0^4}{81A^4} - \frac{\omega_0^2}{81A^2} - \frac{1}{81} - \frac{14\omega_0^2}{81A^2} + \frac{\omega_0^4}{27A^4} + \frac{\omega_0^4}{81A^4}$$

$$- \frac{\omega_0^2}{27A^2} + \frac{\omega_0^4}{81A^4} - \frac{2\omega_0^2}{27A^2} + \frac{1}{81}\Bigg].$$

$$c_4 = \frac{c\pi^4}{18}\left[-\frac{113}{972} - \frac{20\omega_0^2}{27A^2} + \frac{20\omega_0^4}{27A^4}\right] = \underline{\underline{\frac{c\pi^4}{17,496A}\left[720\omega_0^4 - 720\omega_0^2A^2 - 113A^4\right]}}$$

$$v = (3g\tau)^{1/3}\left[1 + \frac{\omega_0}{3\tau} + \frac{A^2-6\omega_0^2}{54\tau^2} + \frac{10\omega_0^3-5\omega_0A^2}{162\tau^3} + \frac{113A^4+720\omega_0^2-720\omega_0^4}{17,496\tau^4}\right].$$

$$\log v = \frac{1}{3}\log 3g\tau + \frac{\omega_0}{3\tau} + \frac{A^2-9\omega_0^2}{54\tau^2} + \frac{3\omega_0^3-\omega_0A^2}{27\tau^3} + \frac{55A^4+486A^2\omega_0^2-729\omega_0^4}{8748\tau^4}.$$

Now $F(\tau+iA) = \frac{1}{3}\log 3g\tau + \frac{\omega_0+iA}{3\tau} + \frac{A^2-9\omega_0^2-18\omega_0 iA}{54\tau^2} +$

$$\frac{5iA^3-A^2\omega_0+9\omega_0^2 iA+3\omega_0^3}{27\tau^3} \quad \frac{13A^4-20A^3\omega_0 i+18A^2\omega_0^2-12A\omega_0 i+36\alpha_4}{36\tau^4}.$$

Hence $55A^4+486A^2\omega_0^2-729\omega_0^4 = (243)(13A^4+18A^2\omega_0^2+36\alpha_4)$.

Thus $\alpha_4 = -\frac{3104A^4+3888A^2\omega_0^2+729\omega_0^4}{8748}$.

$$\phi = \frac{A}{3\tau} - \frac{\omega_0 A}{3\tau} + \frac{5A^3+9\omega_0^2A}{27\tau} - \frac{5A^3\omega_0+3\omega_0^3A}{9\tau} + o(\tau^{-5})$$

$$F(\omega) = \frac{1}{3}\log 3g\omega + \frac{\omega_0}{3\omega} - \frac{9\omega_0^2+8A^2}{54\omega} + \frac{3\omega_0^3+8A^2\omega_0}{27\omega}$$

$$- \frac{729\omega_0^4+3888\omega_0^2A^2+3104A^4}{8748\omega^4} + o(\omega^{-5})$$

$$u(\omega) = (3g\omega)^{1/3}\Bigg[1 + \frac{\omega_0}{3\omega} \quad \frac{3\omega_0^2+4A^2}{27\omega^2} \quad \frac{5\omega_0^3+20\omega_0A^2}{81\omega^3}$$

$$- \frac{248\omega_0^4+1638\omega^2{}_0A^2+1504A^4}{4374\omega^4} + o(\omega^{-5})\Bigg].$$

$$\mu(z) = \frac{c\omega_0\pi}{3A}\, z + \frac{c\pi^2}{54A^2}\,(6\omega_0^2 - A^2)z^2 + \frac{5c\pi^3\omega_0}{162A^3}\,(2\omega_0^2 - A^2)z^3$$

$$+ \frac{c\pi^4}{17{,}496A^4} \cdot (720\omega_0^4 - 720\omega_0^2A^2 - 113A^4)z^4 + \cdots .$$

CHAPTER V.

CONCLUSION

Since the flow $f(\omega)$ can be written as $f_0 + \left(\frac{9\omega^2}{8g}\right)^{1/3}\left[1 + P(\omega)\right]$ where $P(\omega)$ is a power series expansion of ω^{-1}, the entire flow configuration is asymptotic to Γ_a in the sense that $\mathrm{Re}\{f(\omega)\} \to \infty$ and $\mathrm{Im}\{F(\omega)\} \to 0$ uniformly as $\omega \to \infty$ on S. The asymptotic behavior is uniquely determined by g and A in the case of a vertical boundary except for inessential translations in the z and ω planes. Although asymptotic expansions do not always converge, it is possible to get a close approximation for most functions. In this problem we are able to get a close approximation for the flow.

It has been shown that through the methods of differential equations one can reach conclusions achieved by others through methods of integration, composition of functions, and Green's function.

Given similar data, this method can be applied to solve other problems in differential equations. One elementary example is Bessel's equation of integer order:

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

We assume an infinite series expansion $y = \sum_{n=0}^{\infty} a_n x^n$

satisfies the problem. If not, then the method fails.

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Thus the equation becomes $\sum_{n=2}^{\infty} n(n-1)a_n x^n +$

$\sum_{n=1}^{\infty} na_n x^n + (x^2 - p^2) \sum_{n=0}^{\infty} a_n x^n = 0$. For n = 0 we have

$-p^2 a_0 x^0 = 0$. Thus $a_0 = 0$ if p is not zero. For n = 1,

we have $a_1 x - p^2 a_1 x = 0$. Thus $a_1 = 0$ if p is not equal

to ±1. For n = 2, we get $2a_2 x^2 + 2a_2 x^2 + a_0 x^2 - p^2 a_2 x^2 = 0$.

Thus $a_2 = 0$ if p is not equal to ±2. We are saying that p is an integer so we don't have the trivial solution y = 0. Generalizing we get $a_n = 0$ for $n < p$, $a_p, a_{p+2}, a_{p+4}, \cdots$

are nonzero, and $a_{p+1}, a_{p+3}, a_{p+5}, \cdots$ are zero. The solution is thus

$$J_n = \frac{x}{2^n n!} \{1 - \frac{x^2}{2^2 \cdot 1!(n+1)} + \frac{x^4}{2^4 \cdot 2! \cdot (n+1)(n+2)} - \cdots\}.$$

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