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Relationships between Reversible and Connected Automata

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RELATIONSHIPS BETWEEN REVERSIBLE AND CONNECTED AUTOMATA

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INTRODUCTION

Let $\Sigma=\{\sigma_0,\ldots,\sigma_k\}$ be a finite, nonempty collection of symbols. An automaton over the input alphabet Σ is a triple $A=(S,s_0,M)$ where $S=\{s_0,s_1,\ldots\}$ is the nonempty set of states, s_0 is the start state, and $M:S\times\Sigma\to S$ is the state transition function. A is called a finite automaton if S contains a finite number of elements. An automaton may be thought of as a machine which goes from one state into another with each input from Σ.

A nonempty tape is a finite sequence of symbols from Σ. The empty tape is denoted by Λ and the empty set by ϕ. All subscripts and superscripts are non-negative integers. The product of two tapes x and y is the concatenation xy. The product of two sets U and V is $UV=\{uv\colon u\varepsilon U \text{ and } v\varepsilon V\}$. For any tape x let

$$x^0=\Lambda,$$

$$x^1=x,$$

$$x^{n+1}=x^n x^1, \text{ and}$$

$$x^*=\{x^n\colon n \text{ is non-negative}\}.$$

Similarly, if U is a set then

$$U^0=\{\Lambda\},$$

$$U^1=U,$$

$$U^{n+1}=U^nU^1, \text{ and}$$

$$U^*=\{u\colon\ u\varepsilon U^n \text{ and } n \text{ is non-negative}\}.$$

As consequences of the above definitions, $\phi^*=\{\Lambda\}$ and Σ^* is a monoid under concatenation, with identity Λ. If x is a tape and U is a set then $xU=\{x\}U$ and $Ux=U\{x\}$.

The domain of M can be extended from $S X \Sigma$ to $S X \Sigma^*$. For any $s_i \varepsilon S$, $x \varepsilon \Sigma^*$, and $\sigma \varepsilon \Sigma$, let

$$M(s_i,\Lambda)=s_i \text{ and}$$

$$M(s_i,x\sigma)=M(M(s_i,x),\sigma).$$

Let the alphabet $\Sigma_0=\{0,1\}$. Tapes over any other alphabet $\Sigma=\{\sigma_0,\ldots,\sigma_k\}$ can be represented as tapes in Σ_0^*. Define $f(\sigma_i)=10^i$ and for any $x=\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}$ in Σ^*, let

$$f(x)=f(\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j})$$

$$=f(\sigma_{i_1})f(\sigma_{i_2})\ldots f(\sigma_{i_j})$$

$$=10^{i_1}10^{i_2}\ldots10^{i_j}\varepsilon\Sigma_0^*.$$

Now $f:\Sigma^* \to \Sigma_0^*$ is an injection; so any alphabet Σ can be redefined in terms of the alphabet $\Sigma_0=\{0,1\}$. Thus, without loss of generality, only Σ_0 need be considered as an alphabet.

The state transition function M is commonly represented in one of three ways, as in the following example. Suppose $A=(S,s_0,M)$ where $S=\{s_0,s_1,s_2\}$; and $M:S X \Sigma \to S$ is specified by

$$M(s_0,0)=s_0,$$

$$M(s_0,1)=s_1,$$

$$M(s_1,0)=s_2,$$

$$M(s_1,1)=s_0,$$

$$M(s_2,0)=s_0, \text{ and}$$

$$M(s_2,1)=s_2.$$

Then M can be represented

in tabular form,

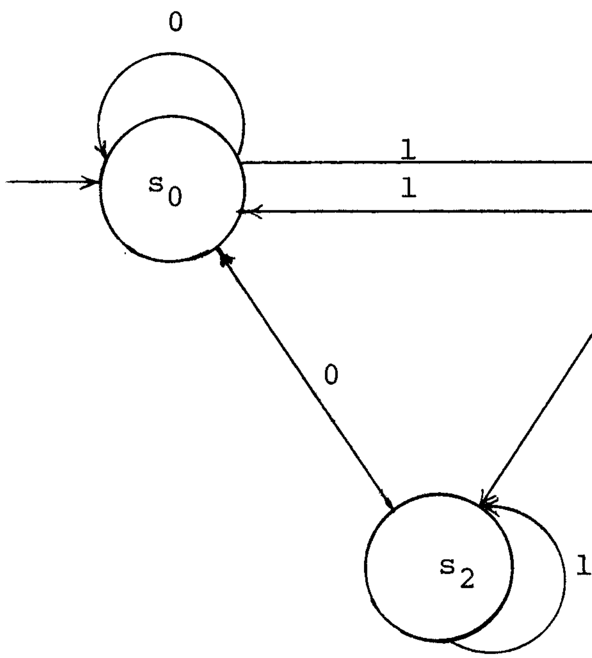
M	0	1
s_0	s_0	s_1
s_1	s_2	s_0
s_2	s_0	s_2 ;

by transition matrices,

$$M(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad M(1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

(In general, if $\sigma\varepsilon\Sigma$ and $M(\sigma)=(a_{ij})$, $a_{ij}=1$ if $M(s_i,\sigma)=s_j$ and $a_{ij}=0$ otherwise.)

and by state diagram,



CHAPTER II

INVESTIGATION

The present investigation explores some of the relationships among special classes of automata defined by Bavel and Muller (1:231-240), Trauth (6:170-175), and Cutlip (2).

Definition 2.1. An automaton $A=(S,s_0,M)$ is said to be reversible if for all $\sigma\varepsilon\Sigma$ there exists $x\varepsilon\Sigma^*$ such that for all $s\varepsilon S$, $M(s,\sigma x)=s$.

Definition 2.2. Automaton $A=(S,s_0,M)$ is called potentially reversible if for all $\sigma\varepsilon\Sigma$ there exists $g:S\rightarrow S$ so that for all $s\varepsilon S$ it is the case that

$$g(M(s,\sigma))=M(g(s),\sigma)=s.$$

Theorem 2.1. If $A=(S,s_0,M)$ is reversible it is potentially reversible.

Proof. For any $\sigma\varepsilon\Sigma$ there exists $x\varepsilon\Sigma^*$ such that for all $s\varepsilon S$, $M(s,\sigma x)=s$. Define

$$g(s)=M(s,x), \text{ yielding}$$

$$g(M(s,\sigma))=M(M(s,\sigma),x)=M(s,\sigma x)=s.$$

Now for all $y\varepsilon\Sigma^*$ there exists $z\varepsilon\Sigma^*$ such that for all $t\varepsilon S$, $M(t,yz)=t$. (Since A is reversible, for each σ_i there exists x_i such that for all $r\varepsilon S$, $M(r,\sigma_i x_i)=r$. If

$$y=\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k} \text{ then } z=x_{i_k}x_{i_{k-1}}\ldots x_{1}.)$$

$$\text{Then } M(g(s),\sigma)=M(g(s),\sigma xy)=M(M(g(s),\sigma x),y)$$

$$=M(g(s),y)=M(M(s,x),y)=M(s,xy)=s.$$

$$\text{Thus } g(M(s,\sigma))=M(g(s),\sigma)=s.$$

Definition 2.3. Automaton $A=(S,s_0,M)$ is partially reversible when for all $\sigma\varepsilon\Sigma, s\varepsilon S$ there exists $x\varepsilon\Sigma^*$ such that for all $t\varepsilon S$ $M(t,\sigma)=s$ implies $M(s,x)=t$.

Theorem 2.2. If $A=(S,s_0,M)$ is reversible then it is partially reversible.

Proof. Let $\sigma\varepsilon\Sigma$ and $s\varepsilon S$ be arbitrary. Then there exists $x\varepsilon\Sigma^*$ so that for all $q\varepsilon S$, $M(q,\sigma x)=q$. If $M(s,\sigma)=q$ then

$$M(s,\sigma x)=M(M(s,\sigma),x)=M(q,x)=s.$$

Definition 2.4. Automaton $A=(S,s_0,M)$ is onto if for all $\sigma\varepsilon\Sigma$, $M(S,\sigma)=S$ where $M(S,\sigma)=\{M(s,\sigma)\colon s\varepsilon S\}$.

Note. A finite automaton with this property is called a permutation machine but this does not necessarily imply a permutation of S in the infinite case. To see this, let $M(s_0,\sigma)=s_0$ and $M(s_i,\sigma)=s_{i-1}$ for all $i>0$.

<u>Theorem 2.3</u>. If $A=(S,s_0,M)$ is potentially reversible then it is onto.

<u>Proof</u>. Let $\sigma\varepsilon\Sigma$ and $s\varepsilon S$ be arbitrary. Then there exists $g:S\rightarrow S$ such that $M(g(s),\sigma)=s$.

<u>Definition 2.5</u>. $A=(S,s_0,M)$ is a unique predecessor automaton if for all $\sigma\varepsilon\Sigma$ and $s,t\varepsilon S$ $M(s,\sigma)=M(t,\sigma)$ implies $s=t$.

<u>Theorem 2.4</u>. If $A=(S,s_0,M)$ is partially or potentially reversible then it is a unique predecessor machine.

<u>Proof</u>. Suppose A is partially reversible. Then let $\sigma\varepsilon\Sigma$ and $s,t\varepsilon S$ be arbitrary. Now if $M(s,\sigma)=M(t,\sigma)$ then there exists $x\varepsilon\Sigma^*$ such that

$$M(M(s,\sigma),x)=s \text{ and } M(M(t,\sigma),x)=t.$$

Therefore $s=t$ and A has the unique predecessor property.

Now let A be potentially reversible and let $M(s,\sigma)=M(t,\sigma)$. Then there exists g such that

$$g(M(s,\sigma))=s \text{ and } g(M(t,\sigma))=t.$$

$$\text{But } g(M(s,\sigma))=g(M(t,\sigma)),$$

so $s=t$; and again A has the unique predecessor property.

Theorem 2.5. When automaton $A=(S,s_0,M)$ is a finite automaton then reversible, partially reversible, potentially reversible, onto, and unique predecessor are equivalent properties.

Proof. By the previous theorems it is sufficient to show that both the onto and unique predecessor properties imply reversibility.

If $A=(S,s_0,M)$ is onto and is not a unique predecessor machine then there exist states s and t such that $s \neq t$ and $M(s,\sigma)=M(t,\sigma)$ for some $\sigma \varepsilon \Sigma$. But then there is $r \varepsilon S$ such that no element is mapped into it under M and σ. This contradicts the onto condition and therefore A is a unique predecessor machine.

Now A onto and unique predecessor implies that for all $s_i \varepsilon S$ there exists an integer $m_i > 0$ such that

$$M(s_i,\sigma^{m_i})=s_i.$$

$$\text{Let } k=\prod_{i=0}^{n-1} m_i \text{ where } S=\{s_0,\ldots,s_{n-1}\};$$

$$\text{then } M(s_i,\sigma^k)=s_i$$

for any $s_i \varepsilon S$ and $i \varepsilon \{0,1,\ldots,n-1\}$. Therefore A is reversible.

Theorem 2.6. If A is an onto and unique predecessor automaton then it is potentially reversible.

Proof. Given $\sigma\varepsilon\Sigma$ and $s\varepsilon S$ let t be the unique predecessor of s under M and σ. Define $g:S\to S$ by $g(s)=t$.

$$\text{Then } M(g(s),\sigma)=M(t,\sigma)=s=g(M(s,\sigma)).$$

Note. As a consequence of theorems 2.3, 2.4, and 2.6; A is onto and has the unique predecessor property if and only if A is such that it is a potentially reversible automaton.

Definition 2.6. $A=(S,s_0,M)$ is connected if for all s S there exists $x\varepsilon\Sigma^*$ such that $M(s_0,x)=s$.

Definition 2.7. An automaton is said to be strongly connected when for all $s,t\varepsilon S$ there exists $x\varepsilon\Sigma^*$ such that $M(s,x)=t$.

Theorem 2.7. If $A=(S,s_0,M)$ is connected and reversible it is strongly connected.

Proof. Let $s,t\varepsilon S$. Then there exist $x,y\varepsilon\Sigma^*$ such that $M(s_0,x)=s$ and $M(s_0,y)=t$. If

$$x=\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$$

then for each σ_{i_j} there exists z_j such that for any $u \varepsilon S$,

$$M(u,\sigma_{i_j} z_j)=u.$$

Then $M(s,z_k z_{k-1} \ldots z_1)=s_0$.

Therefore $M(s,z_k z_{k-1} \ldots z_1 y)=M(s_0,y)=t$.

Theorem 2.8. If $A=(S,s_0,M)$ is strongly connected and has the unique predecessor property then it is partially reversible.

Proof. Let $\sigma \varepsilon \Sigma, s \varepsilon S$, and assume $M(t,\sigma)=s$. Then there exists $x \varepsilon \Sigma^*$ such that $M(s,x)=t$. Since A has the unique predecessor property, there is in fact only one t such that $M(t,\sigma)=s$; so the choice of x depends only on σ and s. Therefore A is partially reversible.

Definition 2.8. $A=(S,s_0,M)$ is called retrievable when for all $\sigma \varepsilon \Sigma$ and $s \varepsilon S$ there exists $x \varepsilon \Sigma^*$ such that $M(s,\sigma x)=s$.

Theorem 2.9. If $A=(S,s_0,M)$ is strongly connected then it is retrievable.

Proof. If $\sigma \varepsilon \Sigma$ and $s \varepsilon S$ there exists $x \varepsilon \Sigma^*$ such that when $M(s,\sigma)=t$ then $M(t,x)=s$. Therefore $M(s,\sigma x)=s$ and A is retrievable.

Theorem 2.10. When $A=(S,s_0,M)$ is retrievable and connected it is strongly connected.

Proof. The proof is identical to the proof given for theorem 2.7.

Theorem 2.11. When machine $A(S,s_0,M)$ is retrievable and has the unique predecessor property it is also partially reversible.

Proof. Let $\sigma\varepsilon\Sigma$ and $s\varepsilon S$ and suppose $M(t,\sigma)=s$. Then there exists $x\varepsilon\Sigma^*$ such that

$$M(t,\sigma x)=t \text{ and}$$

$$M(t,\sigma x)=M(M(t,\sigma),x)=M(s,x)=t.$$

Therefore A is partially reversible.

Theorem 2.12. If an automaton $A=(S,s_0,M)$ is partially reversible then it is retrievable.

Proof. Since A is partially reversible, for all $\sigma\varepsilon\Sigma$, $s\varepsilon S$ there exists $x\varepsilon\Sigma^*$ such that for each $t\varepsilon S$

$$M(t,\sigma)=s \text{ implies } M(s,x)=t.$$

Then for all $t\varepsilon S$ and $\sigma\varepsilon\Sigma$ there is $x\varepsilon\Sigma^*$ such that $M(M(t,\sigma),x)=t$. ($M(t,\sigma)$ plays the role of s in the preceding statement.) Therefore A is retrievable.

Definition 2.9. An automaton $A=(S,s_0,M)$ is called semiretrievable if for all $\sigma\varepsilon\Sigma$ and $s\varepsilon M(S,\sigma)$ there exists $x\varepsilon\Sigma^*$ such that $M(s,x\sigma)=s$.

Theorem 2.13. When $A=(S,s_0,M)$ is retrievable it is semiretrievable.

Proof. Let $\sigma\varepsilon\Sigma$ and $s\varepsilon M(S,\sigma)$. Then $s=M(t,\sigma)$ for some $t\varepsilon S$ and there exists $x\varepsilon\Sigma^*$ such that $M(t,\sigma x)=t$.

Now $M(t,\sigma x)=M(M(t,\sigma),x)=M(s,x)=t$.

But, $M(s,x)=t$ implies $M(M(s,x),\sigma)=M(t,\sigma)$

which yields $M(s,x\sigma)=s$.

Theorem 2.14. If $A=(S,s_0,M)$ is semiretrievable, has the unique predecessor property, and is connected it is strongly connected.

Proof. If s,t S then there exist $x,y\varepsilon\Sigma^*$ such that $M(s_0,x)=s$ and $M(s_0,y)=t$. Now suppose

$$x=\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k};$$

then for each σ_{i_j} $(j<k+1)$ there exists $x_j\varepsilon\Sigma^*$ such that

$$M(s_0,\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_j}x_j)=M(s_0,\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{j-1}}).$$

Then $M(s_0,\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}x_kx_{k-1}\cdots x_1)$

$$=M(M(s_0,\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}x_k),x_{k-1}\cdots x_1)$$

$$=\ldots=M(M(s_0,\sigma_{i_1}),x_1)=M(s_0,\sigma_{i_1}x_1)=s_0.$$

Therefore $M(s,x_kx_{k-1}\cdots x_1y)$

$$=M(M(s_0,x),x_kx_{k-1}\cdots x_1y)$$

$$=M(s_0,y)=t.$$

Therefore A is a strongly connected machine.

Theorem 2.15. If $A=(S,s_0,M)$ is semiretrievable and has the unique predecessor property then it is partially reversible.

Proof. Let $\sigma\varepsilon\Sigma$ and $s\varepsilon S$ be arbitrary. Suppose that, for some $t\varepsilon S$, $M(t,\sigma)=s$. Since A is unique predecessor it is the case that x must take s to t under M. Therefore A is partially reversible.

Definition 2.10. For any tapes $x,y\varepsilon\Sigma^*$ an automaton $A=(S,s_0,M)$ has x and y A-equivalent if $M(s,x)=M(s,y)$ for all $s\varepsilon S$. The set of all tapes A-equivalent to tape x is denoted by $(x)_A$. Define $(x)_A(y)_A(s)=M(s,xy)$.

Each tape $y \varepsilon \Sigma^*$ induces a mapping of S into S, by $y(s)=M(s,y)$. All tapes in $(x)_A$ induce the same mapping of S into S. Define the composition $(x)_A o (y)_A$ by

$$(x)_A o (y)_A = (xy)_A.$$

Definition 2.11. The collection of all sets of A-equivalent tapes is called $\Sigma(A)$.

Theorem 2.16. $A=(S,s_0,M)$ is reversible if and only if $\Sigma(A)$ is a group under functional composition.

Proof. Assume $\Sigma(A)$ is a group. Then for $\sigma \varepsilon \Sigma$ there exists $(y)_A \varepsilon \Sigma(A)$ so that $(\sigma)_A (y)_A = (\Lambda)_A$. Therefore since $(\sigma)_A (y)_A = (\sigma y)_A$ then $M(s,\sigma y)=M(s,\Lambda)=s$ for all $s \varepsilon S$.

Assume now that A is reversible. Then given any $y \varepsilon \Sigma^*$ there exists $x \varepsilon \Sigma^*$ such that for all $s \varepsilon S$, $M(s,yx)=M(s,\Lambda)$. Therefore $(y)_A$ has $(x)_A$ as its inverse. The other group properties are apparent, and $\Sigma(A)$ is a group.

Definition 2.12. An automaton is said to be synchronizable if for some $t \varepsilon S$ there exists $x \varepsilon \Sigma^*$ such that $M(s,x)=t$ for all $s \varepsilon S$.

Theorem 2.17. If $A=(S,s_0,M)$ is reversible and S contains more than one element then A is not a synchronizable automaton.

Proof. If A is synchronizable and S contains more than one element then there exist s,t S where $s \neq t$ and $M(s,x)=M(t,x)$ for some $x \varepsilon \Sigma^*$. Now for some $\sigma \varepsilon \Sigma$,

$$x_1, x_2 \varepsilon \Sigma^* \text{ where } x = x_1 \sigma x_2$$

$$M(s,x_1) \neq M(t,x_1) \text{ but } M(s,x_1\sigma)=M(t,x_1\sigma).$$

This implies that $M(s,x_1\sigma)$ has no unique predecessor under σ. Therefore by theorems 2.1 and 2.4, A is not reversible.

If S contains only one element it is trivially reversible and synchronizable by any $\sigma \varepsilon \Sigma$.

Definition 2.13. An automaton is called state independent when for all $s,t \varepsilon S$ and $x,y \varepsilon \Sigma^*$

$$M(s,x)=M(s,y) \text{ if and only if } M(t,x)=M(t,y).$$

Note. The consequences of this definition differ slightly from those in Trauth (6) due to the inclusion of Λ in Σ^* which is not done by Trauth.

Definition 2.14. Let $A=(S,s_0,M)$; then, given $s \varepsilon S$ and $x,y \varepsilon \Sigma^*$, x is said to be s-equivalent to y if $M(s,x)=M(s,y)$. The set of all tapes s-equivalent to x is denoted by $(x)_s$.

<u>Definition 2.15</u>. The collection of all s-equivalent sets of tapes for all $s \varepsilon S$ is denoted by (Σ^*). (The collection (Σ^*) does not necessarily partition Σ^*.)

<u>Definition 2.16</u>. An automaton that is state independent and is such that (Σ^*) forms a group over its state set is called group-type.

<u>Theorem 2.18</u>. If automaton $A=(S,s_0,M)$ is synchronizable and S contains more than one element then A is not group-type.

<u>Proof</u>. If A is synchronizable and S contains more than one state, then for some $\sigma\varepsilon\Sigma$ and $s,t\varepsilon S$ with $s\neq t$, $M(s,\sigma)=M(t,\sigma)$. But if $s\neq t$ then $M(s,\Lambda)\neq M(t,\Lambda)$ so that A is not state independent and hence not group-type.

If S contains only one element it is trivially group-type and synchronizable.

<u>Theorem 2.19</u>. If $A=(S,s_0,M)$ is group-type then it is reversible.

<u>Proof</u>. Assume A to be group-type. Then since (Σ^*) is a group, for each $(\sigma)_s\varepsilon(\Sigma^*)$ there exists $(x)_s\varepsilon(\Sigma^*)$ such that for some $s\varepsilon S$ $(x)_s$ is the inverse of $(\sigma)_s$. In particular $s=M(s,\Lambda)=(\Lambda)_s=(\sigma)_s(x)_s=M(s,\sigma x)$. Now since A is state independent

$$M(s,\sigma x)=M(s,\Lambda) \text{ if and only if } M(t,\sigma x)=M(t,\Lambda)$$

for all $t \varepsilon S$. Hence A is reversible.

Figure 1 shows an example of an automaton which is reversible by σ^5 but is not state independent since $M(s_0,\sigma^3)=M(s_0,\sigma^6)$ while $M(s_3,\sigma^3)\neq M(s_3,\sigma^6)$.

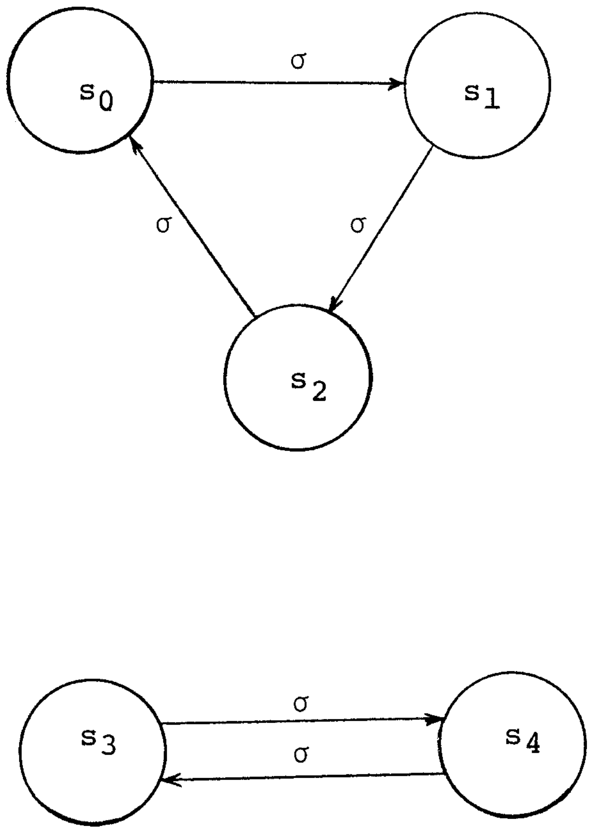


Figure 1

Example of an Automaton

Which is Reversible But is Not State Independent

CHAPTER III

SUMMARY

While it may seem that theorem 2.5 may also hold for the infinite case, Bavel and Muller (1:234-236) give examples to show the independence of those definitions when the state set is not finite. Theorems 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.9, 2.12, 2.13, 2.15, and 2.16 may be found in the same paper.

The definition of state independence is slightly different than in Trauth's work (6:170) since in that paper the input alphabet was extended only to $\Sigma^*-\{\Lambda\}$. This means the definition of a group-type automaton is also different. It was necessary for $\{\Lambda\}$ to be included to obtain the results exhibited in theorems 2.19 and 2.18.

An automaton can be extended from a triple $A=(S,s_0,M)$ to a quadruple $A=(S,s_0,M,F)$ where F is a subset of S called the set of final states.

<u>Definition 3.1</u>. A tape $x\varepsilon\Sigma^*$ is said to be accepted by $A=(S,s_0,M,F)$ if $M(s_0,x)\varepsilon F$.

<u>Definition 3.2</u>. An event is any subset of Σ^*.

<u>Definition 3.3</u>. A regular event is an event that can be expressed using Λ, the symbols of Σ and finitely many applications of union, product, and *.

<u>Definition 3.4</u>. The set of all tapes accepted by $A=(S,s_0,M,F)$ is denoted by $T(A)$.

<u>Theorem 3.1</u>. A subset V of Σ^* is regular if and only if $V=T(A)$ for some finite automaton $A=(S,s_0,M,F)$.

A proof is found in Kleene (3), and will not be given here. McNaughton and Yamada (4) have developed techniques for deriving an automaton that will accept the set of tapes specified by a regular event and for deriving the regular event that will be accepted by a given automaton.

A further topic for investigation might be trying to simplify the methods of construction if it is known in advance that the automaton is reversible or group-type. One could also study the relationship between group-type and reversible automata through the other papers referenced in Trauth (6) and Bavel and Muller (1). It might also be of some interest to see how many of Trauth's results no longer are true using the extended definition of state independence. It was found that a reversible automaton that is connected is also a general repetitive machine as defined by Reynolds (5). There may be a connection between the general repetitive machines as defined by Reynolds and the group-type machines defined here since they are both proper subsets of the class of reversible automata.

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