Summer 1971

A New Confidence Interval for the Mean of a Normal Distribution

David Lee Wallace
Central Washington University

A NEW CONFIDENCE INTERVAL FOR THE MEAN OF A NORMAL DISTRIBUTION

A Thesis

Presented to

the Graduate Faculty

Central Washington State College

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

David Lee Wallace

June, 1971

APPROVED FOR THE GRADUATE FACULTY

____
William B. Owen, COMMITTEE CHAIRMAN

____
David R. Anderson

____
Frederick M. Lister

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Modern statistics may be divided into two broad areas: descriptive statistics and inferential statistics. Descriptive statistics consists of the routine, uninteresting, but necessary tasks of collecting, tabulating, and summarizing data. On the other hand, inferential statistics is the challenging and exciting area concerned with decision making in the face of uncertainty. A typical problem in statistical inference is the following: An experimenter is confronted with a density function $f(x;\theta)$ which describes the underlying population of measurements. The form of f may or may not be known, and θ is a parameter (possibly vector-valued) which describes the population. The statistician's job is to estimate or to test hypotheses about the unknown parameter θ.

From a practical point of view, estimation, and in particular interval estimation, is preferable to hypothesis testing. Suppose $X_1,\ldots,X_n$ is a random sample of size n from $f(x;\theta)$. If we can find two functions f and g such that

$$P[f(X_1,\ldots,X_n) < \theta < g(X_1,\ldots,X_n)] = 1 - \alpha,$$

then f and g are said to determine a $(1 - \alpha)$ 100% confidence interval for θ. The $(1 - \alpha)$ is known as the confidence

<u>coefficient</u>. The interpretation of a confidence interval is as follows. The functions f and g are random variables, and for a given $X_1 = x_1, \ldots, X_n = x_n$, the realization of the random interval

$$[f(x_1,\ldots,x_n),\ g(x_1,\ldots,x_n)]$$

will either contain the unknown parameter or it will not. For any given sample we will not know, but if we repeatedly take random samples of size n and compute a $(1 - \alpha)$ 100% confidence interval for each, we would find that $(1 - \alpha)$ 100% of the intervals so constructed would be correct. In other words, the procedure we use guarantees that we obtain a "correct" interval in $(1 - \alpha)$ proportion of our experiments.

The "goodness" of a confidence interval of a given coefficient is usually measured by its width or, for intervals of random length, expected width. The shorter the width (or expected width) the more informative the interval. Thus, in choosing between two competing 95% confidence intervals, we would usually pick the one having the smallest width.

In this paper, we shall consider interval estimation of the mean of the normal density function which is given by

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\ \exp[-(x-\mu)^2/2\sigma^2] \qquad ,-\infty < x < \infty .$$

Alternatively, we may write $X \sim N(\mu,\sigma^2)$ to indicate the above. The normal density is a function of only two parameters, μ and σ^2. μ is the mean, or center of gravity of the distribution, and σ^2 is the variance, or second moment about the mean of the distribution. The two parameters μ and σ^2 are usually unknown. If they can be estimated accurately, then the experimenter can make valuable and informative statements about his population.

In particular, we shall be concerned with the case when $X \sim N(\mu,\sigma^2)$ with σ known. This situation, while not the most common, does arise in practice. For example, it is well known that I. Q.'s, as measured by properly designed tests, are normally distributed with a variance of about 225. A psychologist may subject a group of children to an intensive study program to see if it has any effect on I. Q. Hence, it would be quite reasonable to assume $X \sim N(\mu, 225)$, and attempt to estimate μ. That is, the mean I. Q. may have changed as a result of the study, but there may be no reason to suspect that the variability has changed. As a second example, consider the diameter of a mass-produced machine part used in an automobile engine. A long history of available data indicated that μ is .00025 and σ is .00002. After many years of use, a quality control inspector notices that more and more of the parts are too big, indicating that perhaps the die has begun to wear. In determining if this is indeed the case, the assumption that $\sigma = .00002$ would be warranted. Thus, even though the mean diameter of the part may have changed, it would be reasonable to assume that the variability of the diameters has remained the same. Many examples can be given of situations in which σ is either known or can be accurately estimated from past data.

CHAPTER II

THEORY

Assume $X \sim N(\mu,\sigma^2)$, where σ known. Since linear combinations of normally distributed random variables are normally distributed, it follows that the sample mean $\overline{X} \sim N(\mu,\sigma^2/n)$. Upon standardization of $\overline{X}$, we see that $Z = \sqrt{n}(\overline{X} - \mu)/\sigma$ is $N(0,1)$. Z will be used exclusively to represent a random variable which is distributed $N(0,1)$. Such a variable Z is said to have a "standard normal" distribution. If we define $z_{\alpha/2}$ by

$$\int_{-\infty}^{z_{\alpha/2}} f(z)dz = 1 - \alpha/2,$$

where $f(z)$ is the density of Z, then $P[Z > z_{\alpha/2}] = \alpha/2$, and from the symmetry of $f(z)$, we also have $P[Z < -z_{\alpha/2}] = \alpha/2$. The only subscript of interest on z is $\alpha/2$. For simplicity of notation, we shall simply write z. Thus,

$$P[-z < Z < z] = 1 - \alpha$$

and since $Z = \sqrt{n}(\overline{X} - \mu)/\sigma$,

$$P[-z < \sqrt{n}(\overline{X} - \mu)/\sigma < z] = 1 - \alpha.$$

After manipulating the inequalities within the probability statement, we see that

$$P\left[\bar{X} - \frac{z\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{z\sigma}{\sqrt{n}}\right] = 1 - \alpha.$$

Hence

$$\left(\bar{X} - \frac{z\sigma}{\sqrt{n}}, \bar{X} + \frac{z\sigma}{\sqrt{n}}\right)$$

is a $(1 - \alpha)$ 100% confidence interval for μ, the unknown mean. The width and expected width of this interval are both $\frac{2z\sigma}{\sqrt{n}}$. This normal interval is optimal in several respects. For example, among all intervals of width $\frac{2z\sigma}{\sqrt{n}}$, the normal interval has maximum confidence coefficient. This interval also possesses the desirable property of unbiasedness. By this we mean that the associated random interval has probability at least as large of capturing the parameter as it has of capturing any other constant.

Consider now the situation where $X \sim N(\mu,\sigma^2)$ and σ is unknown. Let S be the sample standard deviation computed from a random sample of size n. The random variable $T = \frac{(\bar{X} - \mu)\sqrt{n}}{S}$ has a sampling distribution known as Student's t-distribution with a single parameter, $n - 1$, called the degrees of freedom. The density function of T is given by

$$f(t) = \frac{\Gamma(n/2)}{\sqrt{(n-1)\pi}\ \Gamma[(n-1)/2] \left(1 + \frac{t^2}{n-1}\right)^{\frac{n}{2}}}, \quad -\infty < t < \infty.$$

Graphically, the t-distribution is a smooth, symmetric curve resembling the standard normal curve. See Figure 2.1.

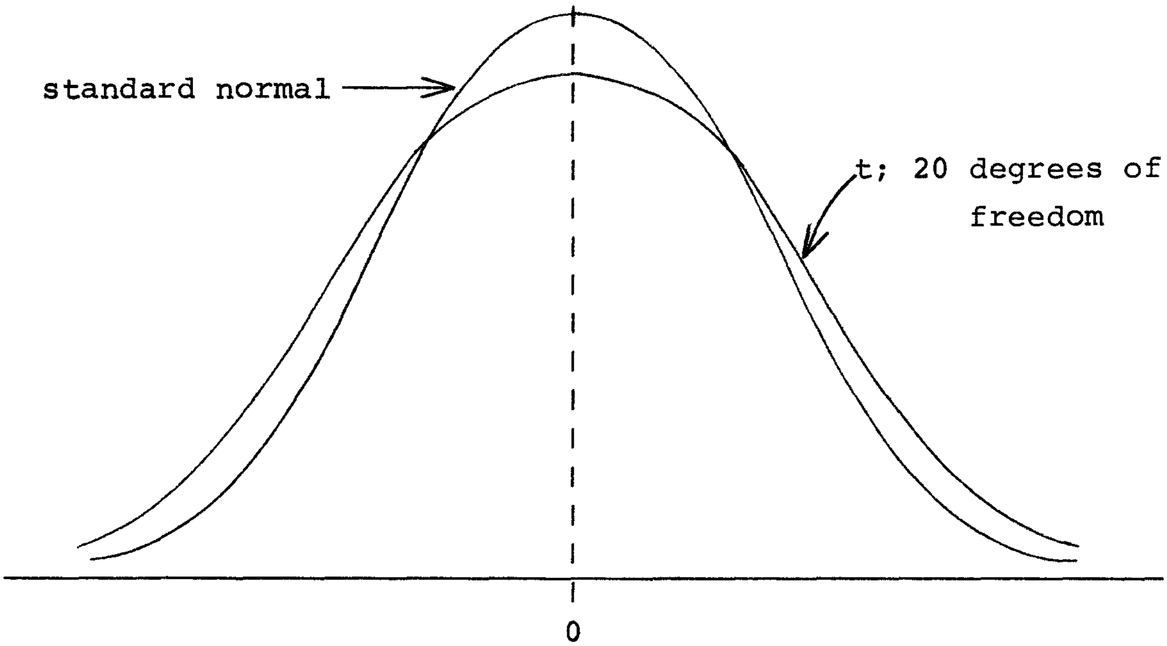

FIGURE 2.1

The t-distribution and the standard normal both have mean zero, but the t density has variance n/(n-2), which is always greater than the unit variance of the standard normal. Define $t_{n-1,\alpha/2}$ by

$$\int_{-\infty}^{t_{n-1,\alpha/2}} f(t)dt = 1 - \alpha/2 .$$

Since only a t-distribution with n-1 degrees of freedom will be used in this paper, and since the only probability level of interest is $\alpha/2$, the subscripts n-1 and $\alpha/2$ will be dropped, and we shall represent $t_{n-1,\alpha/2}$ by t. From the fact that the random variable $T = \frac{\sqrt{n}(\overline{X} - \mu)}{S}$ has a t-distribution, and from the definition of t, we have

$$P\left[-t < \frac{\sqrt{n}(\overline{X} - \mu)}{S} < t\right] = 1 - \alpha,$$

which yields the following:

$$P\left[\overline{X} - \frac{tS}{\sqrt{n}} < \mu < \overline{X} + \frac{tS}{\sqrt{n}}\right] = 1 - \alpha.$$

Hence, when σ is unknown, a $(1-\alpha)$ 100% confidence interval for μ is given by

$$\left(\overline{X} - \frac{tS}{\sqrt{n}}, \overline{X} + \frac{tS}{\sqrt{n}}\right).$$

The width of this "t-interval" is $\frac{2tS}{\sqrt{n}}$ which is random since the sample standard deviation S is a random variable. It can be shown that the expected width is

$$\frac{2\sigma}{\sqrt{n}}\left\{\frac{t\sqrt{2}\,\Gamma(n/2)}{\sqrt{n-1}\,\Gamma[(n-1)/2]}\right\}.$$

For a given n and α, this expected width is greater than the corresponding expected width obtained by using the z-interval with known σ. This result is quite reasonable, as the t variable, $\frac{\sqrt{n}(\bar{X} - \mu)}{S}$, contains an extra element of chance introduced by S, whereas the Z variable, $\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}$, contains only the one random variable $\bar{X}$.

Consider again the situation where $X \sim N(\mu,\sigma^2)$, with σ known. We could ignore the fact that σ is known, and calculate a $(1 - \alpha)$ 100% interval by using the t-distribution and the methods just discussed. Ignoring knowledge of σ may seem like a backward procedure, but since the width of the t-interval is random, it may for any sample be less than the width of the corresponding normal interval. In fact, the t-interval will have smaller width whenever $\frac{2tS}{\sqrt{n}} < \frac{2z\sigma}{\sqrt{n}}$.

Thus, the probability that the t-interval is narrower is

$$p_1 = P\left[\frac{2tS}{\sqrt{n}} < \frac{2z\sigma}{\sqrt{n}}\right].$$

Letting $Y = \frac{(n-1)s^2}{\sigma^2}$ and $L = \frac{(n-1)z^2}{t^2}$, it can be shown that

$$p_1 = P[Y < L],$$

where the random variable Y has a chi-square distribution with parameter n-1. The density function of Y is

$$f_Y(y) = \frac{1}{2^{\frac{n-1}{2}}\,\Gamma[(n-1)/2]}\, y^{\frac{n-3}{2}} e^{\frac{-y}{2}}, \quad 0 < y < \infty.$$

Hence, the probability of a narrower t-interval is

$$p_1 = \int_0^L f_Y(y)dy.$$

Making the transformation y = 2w, we arrive at

$$p_1 = \frac{1}{\Gamma[(n-1)/2]} \gamma[(n-1)/2, L/2] ,$$

where $\gamma(m,a)$ is the incomplete gamma function defined by

$$\gamma(m,a) = \int_0^a x^{m-1}e^{-x}dx.$$

The integral defining p_1 has been evaluated numerically for various sample sizes n and confidence coefficients $1 - \alpha$. The results are presented in Table 2.1.

TABLE 2.1

n \ $1-\alpha$	.995	.99	.975	.95	.90	.75
3	0.0388	0.0649	0.1219	0.1867	0.2708	0.4005
6	0.1148	0.1545	0.2250	0.2856	0.3510	0.4375
9	0.1701	0.2124	0.2759	0.3281	0.3819	0.4505
12	0.2082	0.2484	0.3064	0.3525	0.3991	0.4577
15	0.2359	0.2738	0.3272	0.3689	0.4106	0.4626
18	0.2571	0.2929	0.3425	0.3809	0.4189	0.4661
21	0.2739	0.3079	0.3544	0.3900	0.4252	0.4685
24	0.2878	0.3200	0.3639	0.3973	0.4302	0.4706
27	0.2993	0.3301	0.3718	0.4035	0.4345	0.4726
30	0.3092	0.3387	0.3785	0.4086	0.4380	0.4738

As can be seen from the table, there is more than a slight chance of the t-interval being narrower, and thus better, than the z-interval. This is somewhat unexpected and leads one to consider the following "have your cake and eat it too" procedure: Use the t-interval when it is narrower, and the z-interval when it is narrower. In other words, compute both intervals and use the one having smaller width. The resulting confidence interval is

$$(\bar{X} - W, \bar{X} + W) ,$$

where $W = \min\left(\frac{z\sigma}{\sqrt{n}}, \frac{ts}{\sqrt{n}}\right)$. The width of this new interval, which will be called the min-interval, is $R = 2W$. Since S is a random variable, W is random and hence R is also random. Therefore, the goodness of the min-interval is determined by its expected width, E(R). Notice that since $P[ts < z\sigma] > 0$, E(R) is less than $2z\sigma/\sqrt{n}$, the expected width of the z-interval. More exactly, since $R = 2W$, and W is a function of Y, we may find E(R) by integrating 2W with respect to the density of Y, obtaining

$$\begin{aligned} E(R) &= 2E(W) \\ &= 2\int_0^{\infty} h(y) f_Y(y)\,dy \\ &= \frac{2\sigma}{\sqrt{n}}\int_0^{\infty} \min\left[z,\ t\left(\frac{y}{n-1}\right)^{1/2}\right] f_Y(y)\,dy . \end{aligned}$$

Note that $t\left(\frac{y}{n-1}\right)^{1/2} < z$ if and only if $y < \frac{(n-1)z^2}{t^2} = L$.

Thus,

$$E(R) = \frac{2\sigma}{\sqrt{n}} \left\{ \int_0^L t\left(\frac{y}{n-1}\right)^{1/2} f_Y(y)\,dy + \int_L^\infty z f_Y(y)\,dy \right\}$$

$$= \frac{2\sigma}{\sqrt{n}} \left\{ \int_0^L t\left(\frac{y}{n-1}\right)^{1/2} f_Y(y)\,dy + z\left(1 - \int_0^L f_Y(y)\,dy\right) \right\}.$$

Upon making the transformation $y = 2w$, it follows that

$$E(R) = \frac{2\sigma}{\sqrt{n}} \left\{ \frac{t\sqrt{2}}{\sqrt{n-1}\;\Gamma\left(\frac{n-1}{2}\right)} \gamma\left(\frac{n}{2}, \frac{L}{2}\right) - \frac{z}{\Gamma\left(\frac{n-1}{2}\right)} \gamma\left(\frac{n-1}{2}, \frac{L}{2}\right) + z \right\}.$$

As mentioned previously, E(R) must be less than $\frac{2z\sigma}{\sqrt{n}}$, the width of the normal interval. An evaluation of the expected widths for the normal and min-interval is given in Table 2.2. Note that there are two entries in each cell. The first represents the expected width of the min-interval, and the second the expected width of the normal interval. It can be seen that the min-interval is proportionally better for small sample sizes. This is understandable, since as the sample size increases, there is little difference between the t and z-intervals, and hence little difference between either of them and the min-interval.

TABLE 2.2

n \ 1-α	.995	.99	.975	.95	.90	.75
3	3.1993	2.9095	2.4810	2.1175	1.7182	1.1339
	3.2412	2.9743	2.5881	2.2632	1.8994	1.3282
6	2.2408	2.0374	1.7443	1.5018	1.2376	0.8427
	2.2919	2.1031	1.8301	1.6003	1.3431	0.9392
9	1.8225	1.6594	1.4259	1.2330	1.0219	0.7023
	1.8713	1.7172	1.4943	1.3067	1.0966	0.7669
12	1.5760	1.4366	1.2374	1.0725	0.8916	0.6155
	1.6206	1.4871	1.2941	1.1316	0.9497	0.6641
15	1.4089	1.2854	1.1089	0.9627	0.8017	0.5548
	1.4495	1.3301	1.1574	1.0121	0.8494	0.5940
18	1.2861	1.1742	1.0142	0.8814	0.7349	0.5095
	1.3232	1.2142	1.0566	0.9239	0.7754	0.5423
21	1.1910	1.0879	0.9405	0.8180	0.6826	0.4738
	1.2251	1.1241	0.9782	0.3554	0.7179	0.5020
24	1.1144	1.0184	0.8810	0.7667	0.6402	0.4448
	1.1459	1.0516	0.9150	0.8002	0.6715	0.4696
27	1.0511	0.9609	0.8317	0.7241	0.6050	0.4206
	1.0804	0.9914	0.8627	0.7544	0.6331	0.4427
30	0.9976	0.9122	0.7899	0.6880	0.5751	0.4001
	1.0249	0.9405	0.8184	0.7157	0.6006	0.4200

What about the confidence coefficient associated with the min-interval? It seem intuitively obvious that the confidence coefficient associated with this interval is no longer $1 - \alpha$. We must pay for the reduced width. Bounds for the true confidence coefficient are given in the following theorem.

Theorem 2.1. Let $X \sim N(\mu, \sigma^2)$, σ known. Suppose $X_1, \ldots, X_n$ is a random sample of X. Let

$$P^* = P[\overline{X} - W < \mu < \overline{X} + W] ,$$

where $W = \min\left(\frac{z\sigma}{\sqrt{n}}, \frac{tS}{\sqrt{n}}\right)$. Then

$$1 - 2\alpha < P^* < 1 - \alpha.$$

Proof. Let A be the event "$\overline{X} \pm \frac{z\sigma}{\sqrt{n}}$ contains μ."

Let B be the event "$\overline{X} \pm \frac{tS}{\sqrt{n}}$ contains μ."

Let C be the event "$\overline{X} \pm W$ contains μ."

Note that $P[A] = P[B] = 1 - \alpha$, and that $A \cap B = C$. Hence,

$$P^* = P[C] \leq P[A] = P[B] = 1 - \alpha,$$

which establishes the upper bound. To get the lower bound, we use the additive law of probability, which says that for any two events A and B,

$$P[A \cap B] = P[A] + P[B] - P[A \cup B].$$

Thus,

$$P^* = P[A \cap B] = 1 - \alpha + 1 - \alpha - P[A \cup B]$$
$$= 2 - 2\alpha - P[A \cup B]$$
$$\geq 1 - 2\alpha,$$

since $P[A \cup B] \leq 1$.

The exact confidence coefficient P* may be evaluated as follows:

$$P^* = P[\overline{X} - W < \mu < \overline{X} + W]$$
$$= P\left[\frac{-\min(z\sigma,\ tS)}{\sigma} < Z < \frac{\min(z\sigma,\ tS)}{\sigma}\right].$$

We cannot evaluate the above statement by using the cumulative distribution of Z, because the limits are random variables. Hence, we shall use the fact that if Z and Y are random variables,

$$P[L(Y) < Z < U(Y)] = E_Y P[L(y) < Z < U(y) \mid Y=y].$$

In other words, we shall condition on the value of $Y = \frac{(n-1)S^2}{\sigma^2}$, and then integrate over the density of Y. This gives

$$P^* = \int_0^\infty P\left[\frac{-\min(z\sigma,\ tS)}{\sigma} < Z < \frac{\min(z\sigma,\ tS)}{\sigma} \mid Y\right] f_Y(y)\,dy.$$

$\overline{X}$ and S^2 are independent for a normal density, and since

Y is a function of S^2, it follows that $\bar{X}$ and Y are independent. Thus,

$$P^* = \int_0^\infty P\left[\frac{-\min(z\sigma,\ tS)}{\sigma} < Z < \frac{\min(z\sigma,\ tS)}{\sigma}\right] f_Y(y)\,dy$$

$$= \int_0^\infty \left\{2F_Z\left[\min\left(z,\ \frac{t\sqrt{y}}{\sqrt{n-1}}\right)\right] - 1\right\} f_Y(y)\,dy.$$

Observe that $z < \frac{t\sqrt{y}}{\sqrt{n-1}}$ if and only if $L = \frac{(n-1)z^2}{t^2} < y$.

Thus,

$$P^* = \int_0^L \left[2F_Z\left(\frac{t\sqrt{y}}{\sqrt{n-1}}\right) - 1\right] f_Y(y)\,dy + \int_L^\infty \left[2F_Z(z) - 1\right] f_Y(y)\,dy$$

$$= \int_0^L \left[2F_Z\left(\frac{t\sqrt{y}}{\sqrt{n-1}}\right) - 1\right] f_Y(y)\,dy + (1-\alpha)\left[1 - \int_0^L f_Y(y)\,dy\right]$$

$$= \int_0^L 2F_Z\left(\frac{t\sqrt{y}}{\sqrt{n-1}}\right) f_Y(y)\,dy + (\alpha-2)\int_0^L f_Y(y)\,dy + (1-\alpha). \qquad (1)$$

Since

$$F_Z(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}dx$$

does not have a closed form, the integration in (1) must be performed numerically. For ease of computation and for comparison, $F_Z(z)$ will be approximated by two different methods.

First we consider a simple but reasonably effective approximation. Suppose X_i, i=1,2,3, are independently and identically distributed as a random variable X with probability density function

$$f(x) = 1, \qquad 0 \leq x \leq 1$$
$$f(x) = 0, \qquad \text{elsewhere.}$$

We shall approximate the standard normal variable Z by U, where

$$U = 2 \sum_{i=1}^{3} X_i - 3.$$

U has probability density function g(u), given by

$$g(u) = \frac{(3+u)^2}{16}, \qquad -3 \leq u \leq -1$$
$$g(u) = \frac{3-u^2}{8}, \qquad -1 \leq u \leq 1$$
$$g(u) = \frac{(3-u)^2}{16}, \qquad 1 \leq u \leq 3$$
$$g(u) = 0, \qquad \text{elsewhere.}$$

Integration of g(u) yields G(u), the cumulative distribution function of U:

$$G(u) = 0, \qquad u < -3$$
$$G(u) = \frac{(3+u)^3}{48}, \qquad -3 \leq u \leq -1$$
$$G(u) = \frac{12 + 9u - u^3}{24}, \qquad -1 \leq u \leq 1$$

$$G(u) = 1 - \frac{(3-u)^3}{48}, \qquad 1 \le u \le 3$$

$$G(u) = 1, \qquad 3 < u.$$

$F_Z(z)$ and the approximating function $G(z)$ have been calculated for several values of z, and the results are presented in Table 2.3.

TABLE 2.3

z	G(z)	$F_Z(z)$
0.0	.5000	.5000
0.1	.5375	.5398
0.2	.5747	.5793
0.3	.6114	.6179
0.4	.6473	.6554
0.5	.6823	.6915
0.6	.7160	.7257
0.7	.7482	.7580
0.8	.7787	.7881
0.9	.8071	.8159
1.0	.8333	.8413
1.1	.8571	.8643
1.2	.8785	.8849
1.3	.8976	.9032
1.4	.9147	.9192
1.5	.9297	.9332

z	G(z)	$F_Z(z)$
1.6	.9428	.9452
1.7	.9542	.9554
1.8	.9640	.9641
1.9	.9723	.9713
2.0	.9792	.9772
2.1	.9848	.9821
2.2	.9893	.9861
2.3	.9928	.9893
2.4	.9955	.9918
2.5	.9974	.9938
2.6	.9987	.9953
2.7	.9994	.9965
2.8	.9998	.9974
2.9	.9999	.9981
3.0	1.0000	.9986

Note that the approximation is within .01 even in the worst case. This is indeed remarkable! What we are observing is the convergence to normality of the sum of three independent and identically distributed uniform random variables. This was of course to be expected in view of the Central Limit theorem. What was unexpected, however, is the rapid convergence with only three variables in the sum.

Before substituting G(z) for $F_Z(z)$ in (1), note that when $0 \leq \frac{t\sqrt{y}}{\sqrt{n-1}} \leq 1$, $0 \leq y \leq \frac{(n-1)}{t^2} = L_1$. Therefore if $0 < z \leq 1$, $L \leq L_1$. Upon substituting G(z) for $F_Z(z)$ in (1), we obtain

$$P^* \cong \int_0^L \left[1 + \frac{3ty}{n-1} - \frac{t^3 y^{3/2}}{12(n-1)^{3/2}}\right] f_Y(y)\,dy$$

$$+ (\alpha-2)\int_0^L f_Y(y)\,dy + (1-\alpha),$$

for $0 < z \leq 1$.

When $1 \leq \frac{t\sqrt{y}}{\sqrt{n-1}} \leq 3$, $\frac{(n-1)}{t^2} \leq y \leq \frac{9(n-1)}{t^2} = L_2$.

Thus if $1 \leq z \leq 3$, $L_1 \leq L \leq L_2$ and we obtain

$$P^* \cong \int_0^{L_1} \left\{ 1 + \frac{3ty^{1/2}}{4(n-1)^{1/2}} - \frac{t^3y^{3/2}}{12(n-1)^{3/2}} \right\} f_Y(y)\,dy$$

$$+ \int_{L_1}^{L} \left\{ \frac{7}{8} + \frac{9ty^{1/2}}{8(n-1)^{1/2}} - \frac{3t^2y}{8(n-1)} + \frac{t^3y^{3/2}}{24(n-1)^{3/2}} \right\} f_Y(y)\,dy$$

$$+ (\alpha-2)\int_0^{L} f_Y(y)\,dy + (1-\alpha)$$

for $1 < z \leq 3$.

Finally, when $3 < \frac{t\sqrt{y}}{\sqrt{n-1}}$, $\frac{9(n-1)}{t^2} < y$. Therefore,

if $3 < z$, $L_2 < L$, and we obtain

$$P^* \cong \int_0^{L_1} \left\{ 1 + \frac{3ty^{1/2}}{4(n-1)^{1/2}} - \frac{t^3y^{3/2}}{12(n-1)^{3/2}} \right\} f_Y(y)\,dy$$

$$+ \int_{L_1}^{L_2} \left\{ \frac{7}{8} + \frac{9ty^{1/2}}{8(n-1)^{1/2}} - \frac{3t^2y}{8(n-1)} + \frac{t^3y^{3/2}}{24(n-1)^{3/2}} \right\} f_Y(y)\,dy$$

$$+ \int_{L_2}^{L} f_Y(y)\,dy + (\alpha-2)\int_0^{L} f_Y(y)\,dy + (1-\alpha)$$

for $3 < z$. Table 2.4 shows the evaluation of P^* for certain values of n and the most frequently used values of 1-α. Since the t-distribution converges rapidly to the standard normal distribution for n > 40, only small values of n were used.

TABLE 2.4

n \ 1-α	.995	.99	.975	.95	.90	.75
3	0.9903	0.9809	0.9537	0.9118	0.8366	0.6501
6	0.9937	0.9875	0.9692	0.9399	0.8840	0.7261
9	0.9947	0.9893	0.9734	0.9471	0.8954	0.7431
12	0.9949	0.9898	0.9745	0.9491	0.8986	0.7479
15	0.9950	0.9899	0.9748	0.9497	0.8995	0.7493
18	0.9950	0.9900	0.9750	0.9499	0.8999	0.7498
21	0.9950	0.9900	0.9750	0.9500	0.9000	0.7499
24	0.9950	0.9900	0.9750	0.9500	0.9000	0.7500
27	0.9950	0.9900	0.9750	0.9500	0.9000	0.7500
30	0.9950	0.9900	0.9750	0.9500	0.9000	0.7500

The accuracy of the integration was checked, and found to be correct to five significant digits. Thus the results in Table 2.4 are limited in accuracy only by the approximation for $F_Z(z)$. This suggested looking for a better approximation.

Another approximation for $F_Z(z)$ is given in (1:932):

$$H(z) = 1 - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \left[\frac{a_1}{1+pz} + \frac{a_2}{(1+pz)^2} + \frac{a_3}{(1+pz)^3} \right]$$

for $0 \leq z < \infty$, where $p = .33267$, $a_1 = .4361836$, $a_2 = -.1201676$, and $a_3 = .9372980$. Also, the absolute error in $H(z)$ is less than 1×10^{-5} and hence $H(z)$ is considerably better than $G(z)$ in approximating $F_Z(z)$.

Since $H(z)$ can be used for all values of z greater than 0, we do not need to evaluate P^* piecewise as we did with $G(z)$ in the first approximation. After substituting $H(z)$ for $F_Z(z)$, we have

$$P^* \cong -\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^L \left[\frac{a_1 (n-1)^{1/2}}{(n-1)^{1/2} + pty^{1/2}} + \frac{a_2 (n-1)}{\left[(n-1)^{1/2} + pty^{1/2}\right]^2} + \frac{a_3 (n-1)^{3/2}}{\left[(n-1)^{1/2} + pty^{1/2}\right]^3} \right] \exp[-t^2 y/2(n-1)]\, f_Y(y)\,dy$$

$$+ \alpha \int_0^L f_Y(y)\,dy + (1-\alpha). \qquad (2)$$

The numerical integration in expression (2) has been evaluated numerically on the R.C.A. Spectra 70, using

Simpson's rule with 100 subdivisions per unit. The results are presented in Table 2.5. In most cases, the value of P* computed from 24 partitions per unit was identical to the answer obtained by using 100 partitions per unit. Hence, it appears that the numerical integration is indeed accurate.

TABLE 2.5

n \ 1-α	.995	.99	.975	.95	.90	.75
3	0.9904	0.9810	0.9544	0.9135	0.8401	0.6568
6	0.9908	0.9824	0.9589	0.9232	0.8586	0.6899
9	0.9914	0.9835	0.9616	0.9281	0.8668	0.7025
12	0.9917	0.9842	0.9633	0.9311	0.8715	0.7096
15	0.9920	0.9848	0.9645	0.9331	0.8747	0.7142
18	0.9923	0.9852	0.9654	0.9346	0.8770	0.7175
21	0.9924	0.9855	0.9661	0.9358	0.8787	0.7201
24	0.9926	0.9858	0.9667	0.9367	0.8802	0.7221
27	0.9927	0.9860	0.9671	0.9375	0.8813	0.7237
30	0.9928	0.9862	0.9675	0.9381	0.8823	0.7252

A brief glance at Table 2.5 confirms the fact that P* must lie between $1-2\alpha$ and $1-\alpha$. Roughly speaking, it appears that P* is about halfway between the bounds of Theorem I. Thus, when compared with the normal interval, h is optimal in many respects, the min-interval excels aving smaller expected width. To pay for this we must le for a random and smaller confidence coefficient. question of trade-offs can be resolved by defining riginal optimality criterion.

Definition. The efficiency of a $(1-\beta)$ 100% idence interval is

$$\frac{1 - \beta}{E(W)} \quad .$$

itively, the efficiency of a confidence interval ures the amount of confidence per unit of width, this enables comparison of two intervals with differing idence coefficients. The larger the efficiency, better the interval. Efficiencies have been calted for both the normal interval and the min-interval, these are listed in Table 2.6.

For all sample sizes and confidence coefficients , the min-interval is more efficient than the "opti- normal interval. The complexity of the required uality, however, has precluded a proof of this fact. , note that in Table 2.6 there are two entries in cell. The first is the efficiency of the min-interval the second is the efficiency of the normal interval.

TABLE 2.6

n \ 1-α	.995	.99	.975	.95	.90	.75
3	0.3096 0.3070	0.3372 0.3328	0.3847 0.3767	0.4314 0.4198	0.4889 0.4738	0.5792 0.5646
6	0.4422 0.4341	0.4822 0.4707	0.5497 0.5328	0.6147 0.5936	0.6937 0.6701	0.8187 0.7985
9	0.5440 0.5317	0.5927 0.5765	0.6744 0.6525	0.7528 0.7270	0.8482 0.8207	1.0003 0.9780
12	0.6293 0.6140	0.6851 0.6657	0.7785 0.7534	0.8681 0.8395	0.9775 0.9477	1.1529 1.1293
15	0.7041 0.6864	0.7661 0.7443	0.8698 0.8424	0.9693 0.9386	1.0910 1.0595	1.2871 1.2626
18	0.7715 0.7519	0.8390 0.8153	0.9519 0.9228	1.0604 1.0282	1.1934 1.1607	1.4083 1.3831
21	0.8333 0.8122	0.9059 0.8806	1.0273 0.9967	1.1440 1.1106	1.2874 1.2537	1.5197 1.4939
24	0.8907 0.8683	0.9679 0.9414	1.0972 1.0655	1.2217 1.1872	1.3747 1.3402	1.6233 1.5971
27	0.9445 0.9209	1.0261 0.9986	1.1629 1.1301	1.2946 1.2593	1.4567 1.4215	1.7206 1.6940
30	0.9952 0.9708	1.0811 1.0526	1.2249 1.1913	1.3635 1.3274	1.5342 1.4984	1.8125 1.7856

What are the consequences of these results for the typical experimenter seeking a confidence interval on the mean of a normal density with known variance? By using the method proposed herein, his level of confidence will no longer be exact. For example, if he wants a 95% interval, and if he uses the minimum width interval, his true confidence coefficient P* will be between .90 and .95. This is not a serious deficiency, and in return for sacrificing this small amount of confidence, we gain correspondingly more by reducing the expected width of the interval. That the gain is more than the loss is evidenced by the larger efficiency ratios for the min-interval. A further deficiency of the min-interval is the extra work required in computing both the t and z-intervals to find which gives the minimum. This extra work would no doubt be offset by the experimenter's feeling of supreme satisfaction at legitimately being able to look at the data before deciding which interval to use!

BIBLIOGRAPHY

1. Abramowitz, M. and Stegun, I. A., editors. Handbook of Mathematical Functions. Washington, D. C.: U. S. Government Printing Office, 1964. 1046 pp.

2. Chakravarti, I. M., Laha, R. G., and Roy, J. Handbook of Methods of Applied Statistics, Volume I. New York: John Wiley and Sons, Inc., 1967. 460 pp.