Compartmental Models for Infectious Diseases

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Outline

Mathematical Modeling

- Process of using various mathematical structures to represent real world situations.
- Predict pandemics, natural disasters, population data, and other real world aspects.
- Create and study models.
  - Track a disease's possible spread.
  - Applying real world data.
  - Simulation of a disease and understand behavior.
- Results can be put into perspective to create effective precautions and actions to combat an outbreak.

MEASLES

- Washington State Department of Health
  - Measles, Mumps, Rubella vaccine reminder.
- Snohomish Health District
  - More than one person in a household tested positive for Measles.
- Outbreaks
  - France, Germany, Greece, Italy, Romania
  - 95% of a population should be immunized to prevent an outbreak.[1]

EBOLA

- Democratic Republic of Congo
  - Present since 1976.
  - Outbreak days after last outbreak was declared over.
  - Largely contained.
- Largest epidemic 2014-2016
  - Originated in Liberia, Sierra Leone, and Guinea in West Africa.
  - About 11,000 people died.
- No cure
  - Strict travel restrictions [2].

Ordinary Differential Equations

Brief Overview

- An Ordinary Differential Equation (ODE) is a differential equation containing one or more functions of one independent variable and its derivatives. [3]

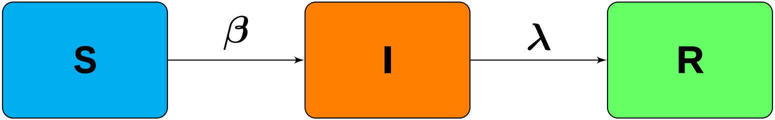
SIR Model

History

- Kermack and McKendrick
  - Earliest classical work on theory of Epidemics.
  - Compartmental models as a technique to simplify Mathematical Modeling of diseases originated in 1927.
- Many choose the SIR to study epidemics.

SIR Model

Brief Overview



- Widely used model
  - Susceptible (S), Infected (I), Recovered (R)
  - Infection rate β .
  - Recovery rate λ.
- Deals with viral diseases
  - Immunity from disease.
- Examples of viral diseases:
  - Measles, Mumps, Chickenpox, and Smallpox. [4]

SIR Model

Assumptions

The assumptions for the basic SIR models are [5]:

1. $S + I + R = 1$
2. The only way an individual can leave the S compartment is to become Infected. The only way an individual can leave the I compartment is to become Recovered.
3. The population is fixed and mixes homogeneously.
4. There is an Infection rate, β.
5. There is a Recovery rate, λ.
6. Once Recovered, an individual is immune and can no longer spread the disease.
7. Age, sex, social status and race do not affect the probability of being Infected.
8. Immunity is not inherited.

SIR Model

More in depth

- Using the stated assumptions we put together the equations:

$$
\begin{aligned}
S'(t) &= -\beta S I \\
I'(t) &= \beta S I - \lambda I \\
R'(t) &= \lambda I
\end{aligned} \tag{1}
$$

- Equilibrium
  - I compartment is at 0.
  - Only one equilibrium.
  - $E_* = (S^*, I^*, R^*)$ where:
    - S^* is anything.
    - $I^* = 0$
    - $R^* = 1 - S^*$

SIR MODEL

MORE IN DEPTH

- Contact number
  - $c = \beta/\lambda$
  - Measures how contagious disease is.
  - Want this to be relatively low.
- Herd immunity
  - Almost the entire population has contracted the disease.
  - There are not enough Susceptible population left to allow an endemic to occur.
- For this type of model the population is fixed, so there is no birth and death rate.

Measles Example

SIR Model

- The Infection rate, $\beta = 0.3$.
- The Recovery rate, $\lambda = 0.2$.
- Contact rate: $c = 1.5$
- Herd immunity: When 95% of the population is immunized an outbreak will be prevented.

Measles Example

Graph

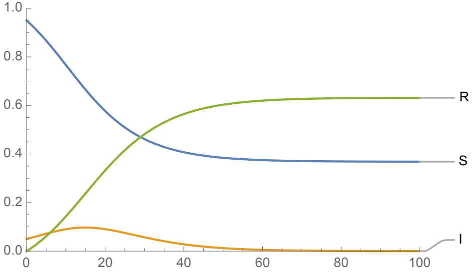


Figure: Mathematica simulation of Measles. Parameters: $S(0) = 0.95$, $I(0) = 0.05$, $R(0) = 0$, $\beta = 0.3$, $\lambda = 0.2$.

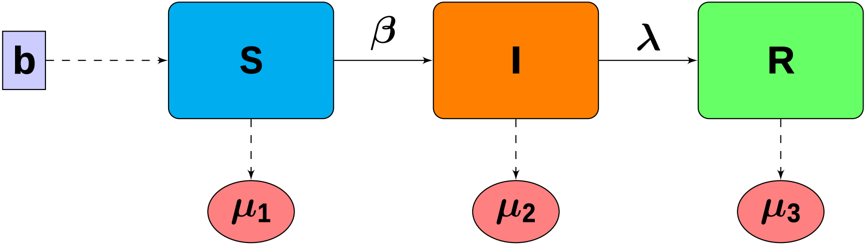
- I fulfills equilibrium point around 65 days.
- $S^* = 0.4$
- $R^* = 0.6$

- Delay Differential Equations (DDE) are commonly used to represent technological and biological control systems.
- Derivative of unknown function at a certain time in terms of the values of the function at previous times.

SIR with Time Delay

Brief Overview

- Compartments
  - Susceptible, Infected, Recovered.
- Variation of a compartmental model.
  - Delay embedded within compartments.
  - Composed of Delay Differential Equations.
- Realistic model
  - Diseases have incubation periods.



SIR WITH TIME DELAY

ASSUMPTIONS

The assumptions for a SIR Model with Time Delay are [5]:

1. Susceptibles must become Infected, Infected must become Recovered.
2. Infection rate, β.
3. Recovery rate, λ.
4. Age, sex, social status and race do not affect infection rate.
5. Immunity is not inherited.
6. Recovered can no longer spread the disease.
7. Delay in time, τ.
8. Birth rate, b.
9. Death rates μ_1, μ_2, μ_3, for S, I, and R compartments respectively, are equal.

SIR WITH TIME DELAY

MORE IN DEPTH

- Using the stated assumptions we put together the equations:

$$\begin{aligned} S'(t) &= -\beta S(t)I(t-\tau) - \mu_1 S(t) + b \\ I'(t) &= \beta S(t)I(t-\tau) - \mu_2 I(t) - \lambda I(t) \\ R'(t) &= \lambda I(t) - \mu_3 R(t) \end{aligned} \tag{2}$$

- Disease-free equilibrium: $E_0 = (S_0, 0, 0)$, where $S_0 = \frac{b}{\mu}$
- Endemic equilibrium: $E_+ = (S^*, I^*, R^*)$, where $S^* = \frac{\mu_2+\lambda}{\beta}$, $I^* = \frac{b-\mu_1 S^*}{\beta S^*}$ and $R^* = \frac{\lambda(b-\mu_1 S^*)}{\mu_3 \beta S^*}$

Ebola Example

SIR Model

- Use SIR with Time Delay Model to track how Ebola can spread.
- Infection rate, $\beta = 0.2$
- Delay in time, $\tau = 1$
- Determine the behavior and simulate of spread Ebola within the population.

Ebola Example

Graph

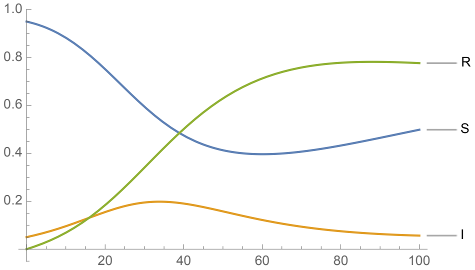


Figure: Mathematica simulation of Ebola. Parameters: $S(0) = 0.95$, $I(0) = 0.05$, $R(0) = 0$, $\beta = 0.2$, $\lambda = 0.1$, $b = 0.013158$, $\mu = 0.00828391$

- Susceptible population decreases quickly.
- Recovered population increases rapidly.
- Infection population gradually increases, then gradually decreases, and plateaus.

SIR With and Without Time Delay

Graph

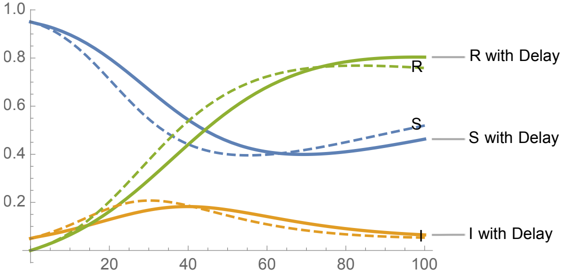


Figure: Mathematica simulation of Ebola with (solid line) and without (dashed line) time-delay. Parameters: $S(0) = 0.95$, $I(0) = 0.05$, $R(0) = 0$, $\beta = 0.2$, $\lambda = 0.1$, $b = 0.013158$, $\mu = 0.00828391$

- Course of time is 100 days, parameters are the same values.
- Time delay
  - Infectious percentage increases later.
    - Affects the S and R compartments.

Airport Data

A Case Study

- Motivated by a potential case in Denver a couple weeks ago.
- Use 2017 airport information with 2014 airline route information to create a network structure [6].
- Travelling while Infected is a common way to spread a disease around the world.
- Model to track and simulate the spread of a disease of an Infected person traveling out of the Seattle Tacoma International Airport (Sea-Tac).
- Modifications:
  1. Weighted average over the number of people in each compartment for all neighboring airports is calculated.

Airport Data

Results

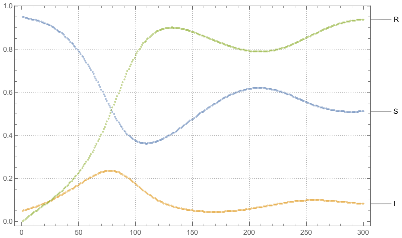
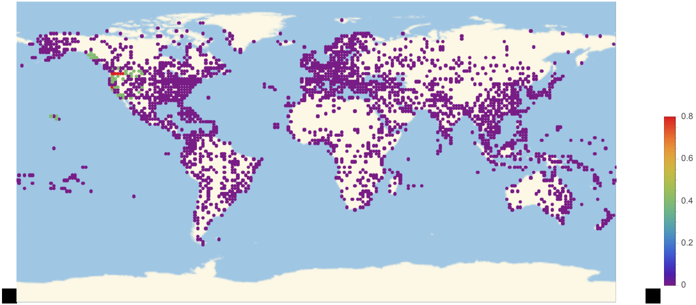


Figure: Mathematica simulation of a disease if it originated in Sea-Tac

- Over a course of 300 days, a disease would not completely die out, so it is endemic.
- In the future, the disease could have another peak or become pandemic.
- Behaviors are similar to the Ebola SIR.
- The Infected population does not outgrow the Recovered and Susceptible populations.

Airport Data

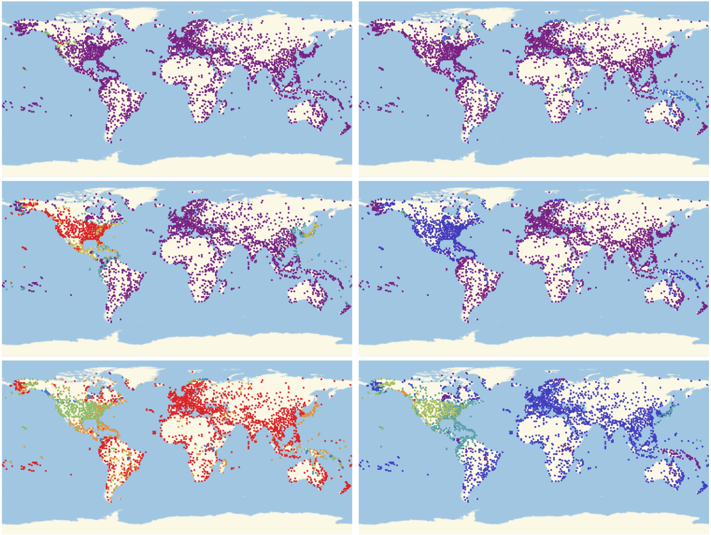
Results



- Visual of how a disease can spread throughout the network structure.
  - Purple: the lowest percentage of Infection.
  - Red: the highest percentage of Infection.

Airport Data

Results



Conclusion

- Studied variations of compartmental models:
  - SIR Model for spread of Measles.
  - SIR with Time Delay Model for spread of Ebola.
  - Used statistics that represented the birth and death rate for the United States.
- Created a simulation for the spread of a disease from Seattle.
  - See how the disease begins as an endemic and becomes a pandemic.

Future Work

Vaccination

- An individual can be granted temporary or permanent immunity.
- Number of studies have been done on how pulse vaccination would be more effective rather than no vaccination or constant vaccination [7, 8].
  - Constant vaccination: A large proportion of newborn population is vaccinated.
  - Pulse vaccination: A fraction of the entire Susceptible class is vaccinated in a pulse every designated amount of years [8].
- With the advancement of modern medicine, new and more effective vaccines become available.
- Researching SIR with vaccine will help health officials decide on what course of action should be taken to get the best possible outcome.

Future Work

Stochastic Epidemic Model

- A Stochastic Model: a collection of random variables.
  - Deals with random behaviors.
- Ideally work well with tracking the spread of a disease using a compartmental model, such as an SIR Model.

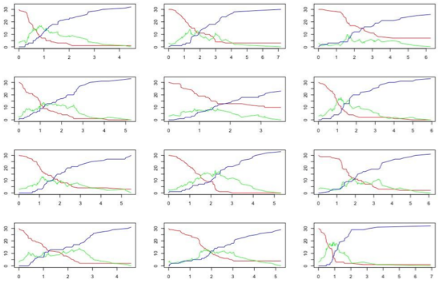


Figure: Gillespie SSA epidemics with the same input parameters[9]

Acknowledgments I

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BIBLIOGRAPHY I

[1] The Daily World. Department of health urges residents to get current on measles vaccination. https://www.thedailyworld.com/news/department-of-health-urges-residents-to-get-current-on-measles-vaccination/, 2018. [Online].

[2] Maggie Fox. Ebola virus breaks out in congo again, days after last one ended. https://www.nbcnews.com/storyline/ebola-virus-outbreak/ebola-virus-breaks-out-congo-again-days-after-last-one-n896531, 2018. [Online].

[3] Nick R. Origin of the term ordinary differential equation. https://hsm.stackexchange.com/questions/5030/what-is-the-origin-of-the-term-ordinary-differential-equations/5032#5032, 2016. [Online].

[4] Herbert W Hethcote. Qualitative analyses of communicable disease models. *Mathematical Biosciences*, 28(3-4): 335–356, 1976.

[5] David Smith and Lang Moore. The sir model for spread of disease. URL https://www.maa.org/press/periodicals/loci/joma/the-sir-model-for-spread-of-disease-the-differential-equation-model.

[6] Vitaliy Kaurov. Modeling a pandemic like ebola with the wolfram language. http://blog.wolfram.com/2014/11/04/modeling-a-pandemic-like-ebola-with-the-wolfram-language/, 2014. [Online].

[7] Boris Shulgin, Lewi Stone, and Zvia Agur. Pulse vaccination strategy in the sir epidemic model. *Bulletin of mathematical biology*, 60(6):1123–1148, 1998.

[8] L Stone, B Shulgin, and Z Agur. Theoretical examination of the pulse vaccination policy in the sir epidemic model. *Mathematical and computer modelling*, 31(4-5):207–216, 2000.

[9] Advisor Dr. Sooie-Hoe Loke Christopher Powell. Epidemic insurance simulations and applications. *SURE 2017 Project Report*, 1(1):8, 2017.